Reading seminar on ∞-categories - plan of talks

Each week, there will be a talk by 1-2 participants on the assigned topic using the provided literature.

We will primarily follow the lecture notes by Martin Gallauer from the University of Warwick (Gallauer, 2023). The lecture notes by Jack Davies (Davies, 2024), Bastiaan Cnossen (Cnossen, 2024) and Yonatan Harpaz (Harpaz, 2019) also contain nice exposition of some of the material. Achim Krause and Thomas Nikolaus from the University of Münster have recorded a neat lecture series on the topic (Krause and Nikolaus, 2020). The authoritative source will be the book by Markus Land (Land, 2021). And last but not least, no survey would be complete without the local Bible by Jacob Lurie (Lurie, 2008).

A very preliminary plan of the talks is the following (it may change until the start of the semester):

0. Introduction: Motivation for considering ∞-categories. Introduction to simplicial sets. Turning topological spaces, resp. categories into simplicial sets via the singular complex, resp. nerve functor. Defining ∞-categories as *weak Kan complexes*, which naturally subsume these notions.

1. Mapping spaces, functor categories: Define the simplicial nerve and show that for a category enriched in Kan complexes, the result is an ∞-category. Use it to define it an ∞-category of (small) ∞-categories and of *spaces*. Hint at the proof that the later are equivalent to ∞-groupoids. Define the mapping space between two objects of an ∞-category and show that it is a space. Sketch a proof that the functors between two ∞-categories form an ∞-category and that a functor is an equivalence $\Leftrightarrow$ it is fully faithful and essentially surjective.
References: (Gallauer, 2023, Section 2.1-2.2), (Krause and Nikolaus, 2020, Lecture 2)

2. Limits and colimits: Define the (co)limits using the usual (co)cone construction. Give explicit descriptions of initial/terminal objects, (co)products, pullbacks, pushouts and mapping telescopes. Sketch the proof that the ∞-category of spaces is (co)complete and that an ∞-category is (co)complete $\Leftrightarrow$ it has finite (co)products and (pushouts)/pullbacks.
References: (Gallauer, 2023, Section 2.3, 3.1, 3.2), (Krause and Nikolaus, 2020, Lecture 3 & 4)

3. Joins, Slice categories, Adjunctions: Define joins and, using them, slice categories. Then state the alternative definition of limits via them. Define the ∞-category BG and show that the homotopy orbits and homotopy fixed points are adjoint functors. Mention Kan extensions in ∞-categories.
References: (Gallauer, 2023, Section 3.3, 3.4, Appendix A.1 & A.2), (Davies, 2024, Section 1.6-1.9), (Krause and Nikolaus, 2020, Lecture 5)

4. Straightening-unstraightening, Yoneda lemma: Sketch the proof that the adjoint functors preserve (co)limits. Define the Yoneda embedding and sketch the proof of the Yoneda lemma. Define the Grothendieck construction for ∞-categories and state the Straightening-unstraightening theorem. Define the twisted arrow category and use it for an alternative description of the Yoneda embedding.
References: (Gallauer, 2023, Section 4.3, 3.4, Appendix A.3),

The content of the following lectures will be based on the interests of attendees, but may concern spectra, stable ∞-categories, E_n-algebras and derived ∞-categories of Abelian categories.

Bibliography

Cnossen, B. (2024) *Introduction to stable homotopy theory*. Available at: https://sites.google.com/view/bastiaan-cnossen/teaching/wi24-introduction-to-stable-homotopy-theory

Davies, J. (2024) *Algebraic Topology II: Stable and Chromatic Homotopy Theory*. Available at: https://sites.google.com/view/jackmdavies/teaching

Gallauer, M. (2023) *∞-categories: a first course*. Available at: https://mgallauer.warwick.ac.uk/teaching/23icats/icats.pdf

Harpaz, Y. (2019) *Little Cube Algebras and Factorization Homology*. Available at: https://www.math.univ-paris13.fr/~harpaz/lecture_notes.pdf

Krause, A. and Nikolaus, T. (2020) *∞-categories and Higher Algebra*. Available at: https://www.youtube.com/watch?v=3IjAy0gHRyY&list=PLsmqTkj4MGTDenpj574aSvIRBROwCugoB

Land, M. (2021) *Introduction to Infinity-Categories*. Birkhäuser Verlag (Compact Textbooks in Mathematics). Available at: https://doi.org/10.1007/978-3-030-61524-6

Lurie, J. (2008) *Higher Topos Theory*. Available at: https://arxiv.org/abs/math/0608040