

Exercises on Spectral Sequences

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Every student learning homological algebra finds out that for a subcomplex of a chain complex $K_\bullet \subseteq C_\bullet$, from the short exact sequence $0 \rightarrow K_\bullet \rightarrow C_\bullet \rightarrow C_\bullet/K_\bullet \rightarrow 0$, one gets a long exact sequence in homology

$$\cdots \rightarrow H_n(K) \rightarrow H_n(C) \rightarrow H_n(C/K) \xrightarrow{\partial} H_{n-1}(K) \rightarrow H_{n-1}(C) \rightarrow H_{n-1}(C/K) \rightarrow \cdots$$

Exercise 1. Recall how the connecting differential ∂ is constructed from the differential of the original complex.

Spectral sequences form a generalization for the situation when we have a (possibly countably infinite) sequence of subcomplexes, called a *filtration*

$$\cdots \subseteq C_\bullet^{(p-1)} \subseteq C_\bullet^{(p)} \subseteq C_\bullet^{(p+1)} \subseteq \cdots \subseteq C_\bullet$$

Being a subcomplex means $\partial(C_\bullet^{(p)}) \subseteq C_\bullet^{(p)}$ for each p .

For example, we may obtain a filtration on the singular complex from a filtration $\cdots \subseteq X^{p-1} \subseteq X^p \subseteq X^{p+1} \subseteq \cdots \subseteq X$ on a space X (like X^p being the p -skeleton).

In practice, computing even the homology of $C_\bullet^{(p)}$ may be difficult and we can access only the homology of the quotient (called *associated graded*)

$$\mathrm{Gr}_\bullet^p := C_\bullet^{(p)} / C_\bullet^{(p-1)}$$

We may pack the long exact sequences for the homologies of all pairs $C_\bullet^{(p)} \subseteq C_\bullet^{(p+1)}$ into a following diagram, where all of the triangles are exact:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_\bullet C^{(p-1)} & \longrightarrow & H_\bullet C^{(p)} & \longrightarrow & H_\bullet C^{(p+1)} \longrightarrow \cdots \\ & & \downarrow & \nearrow \partial & \downarrow & \nearrow \partial & \downarrow \\ & & H_\bullet \mathrm{Gr}^{p-1} & & H_\bullet \mathrm{Gr}^p & & H_\bullet \mathrm{Gr}^{p+1} \end{array}$$

Exercise 2. Suppose we know an element $\alpha \in H_n \mathrm{Gr}^p$. What conditions must be satisfied for it to represent a non-trivial element of $H_n C$?

Even more concisely, we may pack the preceding diagram into a triangle of *bigraded*

$$\text{complexes } H_\bullet C^* := \bigoplus_{n,p} H_n C^{(p)}, H_\bullet \text{Gr}^* := \bigoplus_{n,p} H_n \text{Gr}^p: \quad \begin{array}{ccc} H_\bullet C^* & \xrightarrow{\quad} & H_\bullet C^* \\ \swarrow \partial & & \downarrow \\ & & H_\bullet \text{Gr}^* \end{array}$$

Note that ∂ lowers the homological degree by 1, while the other maps preserve it.

To not worry about the gradings for a moment, we abstract away the situation into a definition, due to Massey.

Definition 3. An *exact couple* is the following diagram of (possibly (bi)graded) complexes, exact at each corner (composition of each consecutive pair of maps is 0):

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & \swarrow \partial & \downarrow p \\ & & E \end{array}$$

Exercise 4. For an exact couple, define its *derived couple* as follows:

- $A' := i(A) \subseteq A$
- $d := p \circ \partial, E' := \ker d / \text{imd}$
- $p'(i(a)) := [p(a)]$
- $\partial'([e]) := \partial(e)$

$$\begin{array}{ccc} A' & \xrightarrow{i} & A' \\ & \swarrow \partial' & \downarrow p' \\ & & E' \end{array}$$

Show that it is well defined and, moreover, again an exact couple.

Definition 5. For a filtered complex, define a **spectral sequence** as a sequence of bigraded complexes $E^1 := H_\bullet \text{Gr}^*$, and $E^{r+1} := (E^r)'$. Moreover, for each r , define the differential d_r on E^r as the d coming from the derived couple. We call E_r the r -th *page*.

Now the spectral sequence machine goes brrrr and we keep on deriving. Each E_{r+1} is a *subquotient* of E_r .

Exercise 6. Unpack the definition as a bigraded complex.

1. We have $E_{n,p}^1 = H_n \text{Gr}^p = H_n(C^{(p)})/C^{(p+1)}$. What does d_1 do on $\alpha \in H_n \text{Gr}^p$?
2. Which bigraded component $E_{n+k,p+\ell}^r$ does $d_r(E_{n,p}^r)$ land in? We call (k, ℓ) its *bidegree*. (Be mindful that other sources may use other grading conventions.)

To obtain something meaningful out of the spectral sequence, we need more assumptions on the filtration. The simplest one is that it is finite, i.e. of the form

$$0 = C_\bullet^{(0)} \subseteq C_\bullet^{(1)} \subseteq \cdots \subseteq C_\bullet^{(k)} = C_\bullet$$

Of course, we may extend it by $C_\bullet^{(p)} = 0$ for $p < 0$ and $C_\bullet^{(p)} = C_\bullet$ for $p > k$.

Exercise 7. Consider the spectral sequence of a finitely filtered complex.

1. For fixed n, p , show that for a sufficiently high r , the source, resp. the target of the differential d_r into, resp. out of $E_{n,p}^r$ are 0. Hence $E_{n,p}^r$ does not depend on r for $r \gg 0$ and we denote it by $E_{n,p}^\infty$.
2. From the long exact sequence coming from the r times derived couple for r large enough, deduce that

$$E_{n,p}^\infty \simeq \text{im}(H_n C^{(p)} \rightarrow H_n C) / \text{im}(H_n C^{(p-1)} \rightarrow H_n C)$$

3. Show that $F_n^p := \text{im}(H_n C^{(p)} \rightarrow H_n C)$ is a filtration on $H_n(C)$ and hence $E_{n,p}^\infty$ is the associated graded of this filtration.

Now we can recover $H_n(C)$ if we can solve the extension problems.

Exercise 8. Show that we can always recover a filtered vector space, up to isomorphism, from its associated graded.

However, for modules over rings which are not fields, we will in general get nontrivial extension problems. Even for vectors spaces, if we worked with cohomology instead of homology, we would get extension problems to recover the multiplicative structure.