Reading seminar on ∞-categories

Remark: In these notes, we will take the liberty of identifying a 1-category D with its nerve (which is an ∞-category). C will always be an ∞-category with finite products.

1. Monoids

In an ∞-category, we would like a monoid to me an object M, along with a product $M \times M \to M$ and a unit $* \to M$ satisfying the associativity and unitality axioms only up to coherences. As before, the machinery to keep track of all the higher coherences is provided by the category Δ.

Definition 1.1: A monoid in C is a functor

$$M : \Delta^{\mathsf{op}} \to C, [n] \mapsto M_n$$

satisfying the *Segal condition*: for every $n \in \mathbb{N}$, the map

$$M_n \to \prod_{i=1}^{n} M_1$$

induced by n inclusions $[1] \hookrightarrow [n]$ is an isomorphism.

We write $\mathsf{Mon}(C) \subset \mathsf{Fun}(\Delta^{\mathsf{op}}, C)$ for the full subcategory of monoids in C.

Remark: In particular, M_0 is the terminal object and the unique map $s_0 : [1] \to [0]$ provides a map

$$e : M_0 \to M_1$$

which we will call the *unit* of M. The inclusion $d_1 : [1] \simeq \{0 < 2\} \hookrightarrow [2]$ provides a map

$$m : M_1 \times M_1 \simeq M_2 \to M_1$$

which we will call the *multiplication* of M. The simplicial identities take care of the unitality and associativity laws.

Definition 1.2: A monoid $M \in \mathsf{Mon}(C)$ is *grouplike* or a *group* in C if the shear map

$$(\mathrm{pr}_1, m) : M \times M \to M \times M, (x, y) \mapsto (x, xy)$$

is an isomorpism. We write $\mathsf{Grp}(C) \subset \mathsf{Mon}(C)$ for the full subcategory of the groups in C.

Proposition 1.1: The loop space functor $\Omega : C_* \to C_*$ extends into a functor $\tilde{\Omega} : C_* \to \mathsf{Grp}(C)$. Abusing the notation, we will call it again Ω.

Definition 1.3: When C admits geometric realizations (i.e. Δ^{op}-indexed limits), the *classifying space* is a functor $\boldsymbol{B} : \mathsf{Mon}(C) \to C_*$ given by

$$M \mapsto \mathrm{colim}_{\Delta^{\mathsf{op}}} M$$

Proposition 1.2: $\boldsymbol{B}$ is the left adjoint to Ω, i.e. there is an adjunction

$$\boldsymbol{B} : \mathsf{Mon}(C) \rightleftarrows C_* : \Omega$$

Theorem 1.1 (Recognition principle): For $C = \mathsf{Spc}$, the preceding adjunction restricts to an equivalence

$$\boldsymbol{B} : \mathsf{Grp}(\mathsf{Spc}) \overset{\simeq}{\rightleftarrows} \mathsf{Spc}_*^{\geq 1} : \Omega$$

between groups in spaces and connected pointed spaces.

Definition 1.4: For $n \in \mathbb{N}$, the E_n-monoid in C is defined inductively:
- $\mathsf{Mon}_{E_0}(C) := C_*$
- $\mathsf{Mon}_{E_n}(C) := \mathsf{Mon}(\mathsf{Mon}_{E_{n-1}}(C)$ for $n > 0$

Analogously for E_n-groups.

Remark: If the monoid laws hold strictly (i.e. when C is a 1-category), by the classical Eckmann-Hilton argument, E_2-monoids are already commutative monoids.

Proposition 1.3: For all $n \in \mathbb{N}$, there is an adjunction

$$\boldsymbol{B}^n : \mathsf{Mon}_{E_n}(\mathsf{Spc}) \rightleftarrows \mathsf{Spc}_* : \Omega^n$$

which restricts to an equivalence

$$\boldsymbol{B}^n : \mathsf{Grp}_{E_n}(\mathsf{Spc}) \overset{\simeq}{\rightleftarrows} \mathsf{Spc}_*^{\geq n} : \Omega^n$$

between E_n-groups in spaces and $(n-1)$-connected spaces (with trivial π_i for $i < n$).

2. Commutative monoids

To define the notion of monoids commuting up to higher coherences, we need to "add more" structure to the category Δ taking care of the symmetries.

Definition 2.1: Len Fin_* be the (skeletal) 1-category of finite pointed sets. Abusing the notation, we will call its elements again $[n]$ for $n \in \mathbb{N}$.

Remark: Alternatively, Fin_* can be considered as a category of finite sets and partially defined maps, with the base point $*$ acting as a "trash bin" where every element whose image is not defined is mapped.

There is an inclusion $\Delta \hookrightarrow \mathsf{Fin}_*^{\mathrm{op}}$ taking an order preserving map f to a partial map assigning to every element the maximum of its preimage in f.

Definition 2.2: A commutative monoid in C is a functor

$$M : \mathsf{Fin}_* \to C, [n] \mapsto M_n$$

satisfying the *Segal condition*: for every $n \in \mathbb{N}$, the map $M_n \to \prod_{i=1}^n M_1$ induced by n inclusions $[1] \hookrightarrow [n]$ is an isomorphism.

We write $\mathsf{CMon}(C) \subset \mathsf{Fun}(\mathsf{Fin}_*, C)$ for the full subcategory of monoids in C.

Proposition 2.1: There is an adjunction

$$\boldsymbol{B}^\infty : \mathsf{CGrp}\ (\mathsf{Spc}) \rightleftarrows \mathsf{Sp}_* : \Omega^\infty$$

between the commutative groups in spaces an spectra, which restricts to an equivalence on the full subcategory $\mathsf{Sp}^{\geq 0}$ of *connective* spectra. (A spectrum X is called connective if its n-th space X_n is $(n-1)$-connected for every n).

3. Monoidal categories

Definition 3.1: A monoidal category is a monoid in Cat_∞.

A symmetric monoidal category is a commutative monoid in Cat_∞.

Remark: By straightening-unstraightening, a monoidal, resp. symmetric monoidal category is equivalently given by a cocartesian fibration $C^\otimes \to \Delta^{\mathsf{op}}$, resp. $C^\otimes \to \mathsf{Fin}_*$.