Reading seminar on infinity-categories

1. Stable ∞-categories

Definition 1.1: An ∞-category is called *pointed* if there is an object which is both initial and terminal. This object will be called the *zero object* and denoted 0.

Example: For an ∞-category C with a terminal object $*$, the ∞-category C_* is pointed.

Definition 1.2: A *null sequence* in a pointed ∞-category is a commutative square of the form

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow{\scriptstyle f} \\ 0 & \longrightarrow & Z \end{array}$$

It is called *fiber* sequence if it is a pullback (and X is the *fiber* of f), *cofiber* sequence if it is a pushout (and Z is the *cofiber* of g) and *exact* sequence if it is both.

Definition 1.3: For an object X in a pointed ∞-category C, the *loop space* ΩX is the fiber of $0 \to X$ and the *suspension* ΣX is the cofiber of $X \to 0$. They assemble into functors

$$\Omega, \Sigma : C \to C$$

Remark: These notions are not interesting for ordinary categories - they allways yield 0 there.

Definition 1.4: A pointed ∞-category is called *stable* if all pushouts and pullback exists and, moreover, a commutative square is a pushout square if and only if it is a pullback square.

Proposition 1.1: For a pointed ∞-category C, the following are equivalent:
1. C is stable
2. all fiber and cofiber sequences in C are exact
3. C admits fibers and the functor $\Omega : C \to C$ is an equivalence
4. C admits cofibers and the functor $\Sigma : C \to C$ is an equivalence

Definition 1.5: A pointed ∞-category is *semiadditive* if it admits all finite products and coproducts and for any $X, Y \in C$, the map

$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix} : X \sqcup Y \to X \times Y$$

is an equivalence. The (co)product is then called *biproduct* and denoted $X \oplus Y$.

Moreover, it is *additive* if the *shear map*

$$\begin{pmatrix} \mathrm{id}_X & \mathrm{id}_X \\ 0 & \mathrm{id}_X \end{pmatrix} : X \sqcup X \to X \times X$$

is an equivalence

Proposition 1.2: Stable ∞-categories are additive.

2. Spectra

Definition 2.1: Let C be an ∞-category admitting finite limits. Its *stabilization* $\mathsf{Sp}(C)$ is the $\mathbb{N}^{\mathrm{op}}$-indexed limit in the (very large) ∞-category of ∞-categories

$$\lim\left(\dots \overset{\Omega}{\to} C_* \overset{\Omega}{\to} C_* \overset{\Omega}{\to} C_*\right)$$

Its objects are called *spectra* of C. The stabilization of the ∞-category of spaces will be simply called *spectra* and denoted Sp.

In particular, a spectrum X of C is a collection of objects $X_n \in C, n \in \mathbb{N}$, along with structure maps

$$\sigma_n : X_n \overset{\simeq}{\to} \Omega X_{n+1}$$

A morphism f of spectra X and Y is a collection of maps $f_n : X_n \to Y_n$, along with a choice of homotopies filling the squares

$$\begin{array}{ccc} X_n & \xrightarrow{\simeq} & \Omega X_{n+1} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ Y_n & \xrightarrow{\simeq} & \Omega Y_{n+1} \end{array}$$

Theorem 2.1: For every ∞-category C with finite limits, its stabilization $\mathsf{Sp}(C)$ is stable.