Reading seminar on infinity-categories

1. Yoneda

Definition 1.1: For an ∞-category C, let $\mathcal{P}(C)$ be the functor ∞-category $\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Spc})$, called the *presheaves* on C.

Theorem 1.1: For a (small) ∞-category C, there is a fully faithful functor

$$y : C \to \mathcal{P}(C)$$

such that for each $F \in \mathcal{P}(C)$ and $c \in C$, there is a homotopy equivalence

$$\mathsf{Map}_{\mathcal{P}(C)}(y(c), F) \simeq F(c)$$

Proof sketch: We cannot straighforwardly generalize the case of ordinary categories, as the composition is defined only up to a contractible choice.

We will use the fact that Spc is a homotopy coherent nerve of a simplicial category, where we have a canonical choice of composition. Consider the following composite of simplicial categories

$$\mathfrak{C}[C^{\mathsf{op}} \times C] \to \mathfrak{C}[C]^{\mathsf{op}} \times \mathfrak{C}[C] \overset{\mathsf{Hom}_{\mathfrak{C}[C]}}{\to} \mathsf{sSet} \overset{\mathsf{Sing} \circ |-|}{\to} \mathsf{Kan}$$

By adjunction, this corresponds to a functor

$$C^{\mathsf{op}} \times C \to N(\mathsf{Kan}) = \mathsf{Spc}$$

and by the simplicial hom adjunction to a functor

$$y : C \to \mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Spc}) = \mathcal{P}(C)$$

There is a Yoneda lemma for simplicially enriched categories, the proof is the same as in the classical case and it is fully faithful, which induced the fully faithfulsness of y.

The homotopy equivalence can similarly be translated into simplicial categories, however, one needs to express $\mathcal{P}(C)$ as a homotopy coherent nerve of a simplicial category. □

Remark: $y : C^{\mathsf{op}} \times C \to \mathsf{Spc}$ is pointwise equivalent to the mapping spaces, but not *equal*.

1.1. Straightening / unstraightening

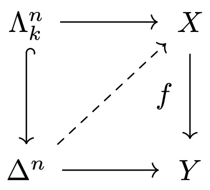
A functor $\pi : C \to D$ between ordinary categories is called *left fibered* in sets if for every $f : \pi(x) \to y$, there is a unique lift $\tilde{f}$ with $\pi\left(\tilde{f}\right) = f$. There is a correspondence

$$\mathsf{LFib}\ (D) \simeq \mathsf{Fun}(D, \mathsf{Set})$$

where the set on the left is the set of the left fibered functors over D.

A similar situation is true for ∞-categories.

Definition 1.1.1: Consider a map of simplicial sets $f : X \to Y$ and the following lifting problem:



A map is called	if $\exists$ a lift $\forall$ n and k equal
fibration	$0, ..., n$
left fibration	$0, ..., n-1$
right fibration	$1, ..., n$
inner fibration	$1, ..., n-1$

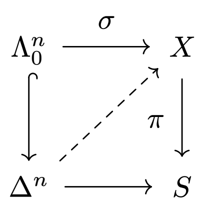
The ∞-categories of left, resp. right fibrations over Y will be denoted $\mathsf{LFib}(Y)$, resp. $\mathsf{RFib}(Y)$.

Proposition 1.1.1 (Straightening-unstraightening, discrete): For a (small) ∞-category C, there are equivalences of ∞-categories

$$\mathsf{Fun}(C, \mathsf{Spc}) \simeq \mathsf{LFib}(C)$$
$$\mathcal{P}(C) = \mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Spc}) \simeq \mathsf{RFib}(C)$$

Similarly, for there is a correspondence between *pseudofunctors* with the target in ordinary categories and the Grothendieck opfibration. This result has been extended by Lurie as well.

Definition 1.1.2: Let $\pi : X \to S$ be a map of simplicial set and $f : x \to y$ an edge in X. We say that f is π-cocartesian if there exists a lift in every diagram



where σ maps $\Delta^{\{0,1\}}$ to f (and π-cartesian for diagrams where σ maps $\Delta^{\{n-1,n\}}$ to f).

π is called *(co)cartesian fibration* if it is an inner fibration and for every $x \in X$ and every edge $f : \pi(x) \to y$, there is a π-(co)cartesian edge $\tilde{f} : x \to \tilde{y}$ with $\pi\left(\tilde{f}\right) = f$.

Theorem 1.1.1 (Grothendieck-Lurie correspondence): There are equivalences

$$\mathsf{Fun}(C, \mathsf{Cat}_\infty) \simeq \mathsf{Fib}^{\mathrm{coc}}(C)$$
$$\mathsf{Fun}(C^{\mathsf{op}}, \mathsf{Cat}_\infty) \simeq \mathsf{Fib}^{\mathrm{car}}(C)$$

with the classes of (co)cartesian fibrations on the right hand side.