Reading seminar on ∞-categories

1. Naïve attempts at a definition

You may have noticed that for two ordinary categories, functors between them form another category (with morphisms given by natural transformations). So in the category of categories Cat, apart from morphisms (functors), we have some 2-morphisms between morphisms.

This suggests a construction of higher categories by allowing the morphisms between two objects to have a different structure than a mere set. Such a notion is called *enrichment*.

Definition 1.1: Let $\mathcal{E}$ be a category with a (categorical) product $\times$ and a terminal object $1_{\mathcal{E}}$. A category enriched over $\mathcal{E}$ is a collection of objects C and an object $\operatorname{Hom}(a, b)$ in $\mathcal{E}$ for each $a, b \in C$, along with morphisms in $\mathcal{E}$

$$\operatorname{Hom}(a, b) \times \operatorname{Hom}(b, c) \to \operatorname{Hom}(a, c)$$
$$1_{\mathcal{E}} \to \operatorname{Hom}(a, a)$$

for each $a, b, c \in C$, satisfying appropriate associativity and unit axioms.

The definition is very analogous to the definition of a category, with hom sets replaced by objects of $\mathcal{E}$. For example, a category enriched over Cat has hom categories.

Now, it is natural to make the following inductive definition.

Definition 1.2: Cat_1 are ordinary categories. Cat_{n+1} are categories enriched over Cat_n. Cat_n are called *strict* n-categories.

Are we done at this point? Unfortunately, this definition doesn't capture most higher categorical notions that arise in nature. The problem is that by having $(n + 1)$-morphisms, the laws between n-morphisms such as associativity and unitality can be expressed not as equalities, but as specific $(n + 1)$-morphisms witnessing them. But then, these should have $(n + 2)$-morphisms witnessing their laws, and so on.

Exercise: Show that given 2-morphisms $(a \circ b) \circ c \Rightarrow a \circ (b \circ c)$ witnessing the associativity of composition of 1-morphisms, there are two different ways to map $((a \circ b) \circ c) \circ d$ to $a \circ (b \circ (c \circ d))$. We would like to have a 3-morphism witnessing an equivalence between these.

What we really need is a notion of *weak* n-categories, where laws hold only up to a higher equivalence. For low n, they have been explicitly defined, but the lists of axioms quickly undergo a combinatorial explosion.

Already in case $n = 3$, this notion is more general than the one defined above.

Example: For a topological space X, let its fundamental 3-groupoid $\pi_{\leq 3} X$ be a weak 3-category whose objects are points of X, morphisms are paths between points, 2-morphisms are homotopies between paths and 3-morphisms are homotopies between homotopies (factored up to a higher homotopy, i.e. actually equivalence classes of homotopies).
The fundamental 3-groupoid of a 2-sphere $\pi_{\leq 3} \mathbb{S}^2$ is not equivalent to any strict 3-category.

How is it possible to tackle this combinatorial explosion of *coherence laws*? Alexander Grothendieck suggested that they should be modeled by objects we already (a little bit) know how to study in mathematics, namely topological spaces.

Homotopy hypothesis: ∞-groupoids (weak ∞-categories with all morphisms invertible) are equivalent to (homotopy types of) topological spaces.

In constructing various theories of ∞-categories, the homotopy hypothesis was seen as a guiding principle that a good theory should satisfy.

In these lectures, we will only concern ourselves with the theory of $(\infty, 1)$-categories, i.e. categories which have all k-morphisms invertible for $k > 1$. While not being the most general case, on one hand they suffice for most applications and on the other, they can be used as a basic building block for the theory of (∞, n)-categories, with $n = 2, \ldots, \infty$. In light of this, from now on, by ∞-categories we will actually mean $(\infty, 1)$-categories.

Since $(\infty, 1)$-categories have all higher morphisms invertible, the hom objects in them should be ∞-groupoids. The homotopy hypothesis immediately suggests the following definition.

Definition 1.3: A *topological category* is a category enriched over topological spaces.

While this definition provides a sensible notion of $(\infty, 1)$-categories, it is actually not often used in practice, the reason being similar to before. Most natural constructions yield composition of morphisms that is associative only up to a homotopy, not "on the nose", as the definition of an enriched category demands. It turns out that in this case, it is possible to introduce a straightening procedure, which turns composition associative up to homotopy into a strictly associative one, however that comes at the cost of technical complexity. Issues also arise when defining functors between such categories: it turns out one has to consider all the functors between topologically enriched categories that are in a certain sense "equivalent" to the source, resp. target.

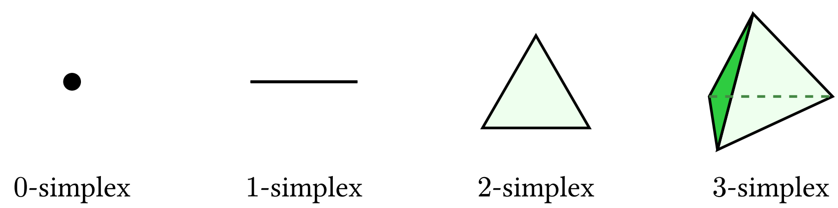
In practice, other definitions are used. However, all of them can be shown to be equivalent to topological categories in a precise sense.

We will study the most common one, called *quasicategories* or *weak Kan complexes*. As a first step, we will replace topological spaces with a combinatorially better tractable structure called *simplicial sets*, which can nevertheless model the same homotopy theory.

2. Three faces of simplicial sets

2.1. Geometric

Definition 2.1.1: A (geometric) n-simplex is a convex hull of $n + 1$ affinely independent points in an affine space.



Remark: For concreteness, the standard n-simplex is often defined as $\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_0 + \ldots + t_n = 1\}$, i.e. the convex hull of the canonical basis vectors of $\mathbb{R}^{n+1}$.

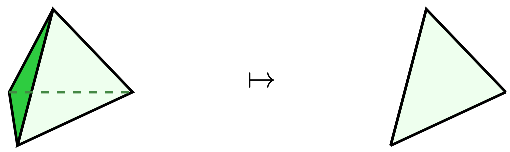
A simplicial set can be imagined as a collection of n-simplices (in a sense the simplest n-dimensional objects) for various n, along with prescribed ways of gluing them. All of this data, however, is given abstractly.

2.2. Combinatorial

More generally, we may consider n-simplex to be a convex hull of a list of points $(a_0, ..., a_n)$, where the set of distinct points is affinely independent, but some points may occur more times; if that happens, such a simplex will be called *degenerate*. The ordering of the list naturally provides an orientation for the simplex.

It is clear that the boundary of an n-simplex consists of k-simplices for $k < n$, which are convex hulls of subsets of its defining points.

Definition 2.2.1: For an n-simplex x and $i = 0, ..., n$, let $\delta^i x$ be the convex hull of all of its defining points, except the i-th. It is called the i-th face of x.



From an n-simplex, we can form a degenerate $(n + 1)$-simplex by duplicating some of its points.

Definition 2.2.2: For an n simplex x given by a list of points $(a_0, ..., a_n)$ and $i = 0, ..., n$, let $\sigma^i x$ be the convex hull of the points $(a_0, ..., a_{i-1}, a_i, a_i, a_{i+1}, ..., a_n)$.

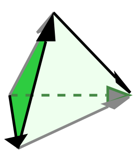
To unchain ourselves from using concrete geometric representations, we would like to present a simplicial set X as sets of "n-simplices" X_n for all n, along with boundary and degeneracy data. For that, it better satisfy the relations that hold between geometric simplices, such as

1. $\delta^i \delta^j = \delta^{j-1} \delta^i$ if $i < j$
2. $\delta^i \sigma^j = \sigma^{j-1} \delta^i$ if $i < j$
3. $\delta^i \sigma^j = \text{id}$ if $i = j$ or $i = j + 1$
4. $\delta^i \sigma^j = \sigma^j \delta^{i-1}$ if $i > j + 1$
5. $\sigma^i \sigma^j = \sigma^{j+1} \sigma^i$ if $i \leq j$

In fact, all relations follow from the ones above. To not having to remember and carry them all along the way, instead, we will construct an algebraic structure that encodes them.

2.3. Algebraic

When the simplices are given orientation, they also happen to be convenient objects to represent (higher) categorical composition in the following sense. Consider a simplex which is the hull of points $(a_0, ..., a_n)$, view these points as objects and its edges between points a_{i-1} and a_i as morphisms $a_{i-1} \to a_i$. Then the other 1-simplices in the boundary correspond to their various compositions, 2-simplices to 2-morphisms witnessing the composition and so on.



This suggest a construction of an abstract category, with the morphisms corresponding to face and degeneracy maps. Due to conventions, the maps will go in the opposite direction.

Definition 2.3.1: For $n \in \mathbb{N}$, let $[n]$ be a category with $n + 1$ objects $\{0, ..., n\}$ and morphisms generated by the chain

$$0 \leftarrow 1 \leftarrow ... \leftarrow n$$

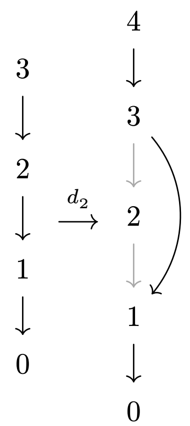
Let Δ be the category with objects $[n]$ for $n \in \mathbb{N}$ and morphisms the functors between these categories. It is called the *simplex category*.

Remark: Equivalently, we may define $[n]$ as the finite linearly ordered set $\{0 < ... < n\}$ and the morphisms of Δ as the order preserving maps between them. We chose the preceding definition to emphasize that $[n]$ is the 1-categorical variant of an n-simplex (its non-identity morphisms form a 1-boundary of an n-simplex).

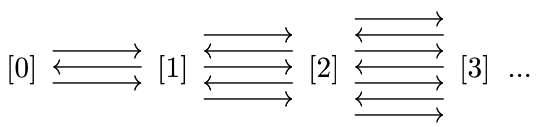
Of particular importance are the morphism representing face and degeneracy maps.

Definition 2.3.2: Fix $n \geq 0$. Let $d_{i,n} : [n] \to [n + 1]$ for $i = 0, ..., n + 1$ be the morphism of Δ given by the inclusion of objects which misses the object i. It is called a *face map*.

Let $s_{i,n} : [n + 1] \to [n]$ for $i = 0, ..., n$ be the morphism of Δ given by the surjection on objects which maps i and $i + 1$ to the same object. It is called a *degeneracy map*.



We will write just d_i, s_i as n will be clear from the context. In fact, any map in Δ can be factored as a composition of face and degeneracy maps. Using these generating morphisms, Δ is often drawn in the following way, with the rightward morphisms being face maps and leftward degeneracies.



Definition 2.3.3: A simplicial set is a functor $X : \Delta^{\text{op}} \to \text{Set}$ to the category of sets. We will denote by X_n its value on the object n and call its elements the (abstract) n-simplices of X.

Definition 2.3.4: Let Δ^n be the representable simplicial set given by $\Delta^n_k = \mathrm{Hom}_\Delta([k], [n])$. It is the simplicial set freely generated by one non-degenerate abstract n-simplex,

Remark: For a simplicial set X, by the Yoneda lemma, the n-simplices X_n correspond to natural transformations $\mathrm{Nat}(\Delta^n, X)$.

A simplicial set can be seen as a collection of simplices along with gluing data, i.e. maps showing which simplex is a face or a degeneracy of which. This is made explicit using *geometric realization.*

Definition 2.3.5: Let $\underline{\Delta^n}$ be the standard geometric n-simplex. For a simplicial set X, its *geometric realization* is the topological space $|X| = \left(\bigsqcup_n \underline{\Delta^n} \times X_n\right) / \sim$, with the equivalence relation given by $(\delta^i a, x) \sim (a, X(d_i)x)$ and $(\sigma^i a, x) \sim (a, X(s_i)x)$ for all face and degeneracy maps.

We will now show that simplicial sets subsume two notions that we would like to generalize with ∞-categories and that are in a way the extreme cases. On one hand, topological spaces have non-trivial n-morphisms for each n (homotopies), but they are all invertible (by inverse homotopies). On the other hand, categories only have non-trivial 1-morphisms, but they are non-invertible.

Although the constructions are very similar, they are by historical conventions given rather ad-hoc sounding names.

2.4. Simplicial sets from topological spaces

Definition 2.4.1: For a topological space S, its *singular complex* is given by

$$\mathrm{Sing}\,(S)_n = \mathrm{Hom}_{\mathrm{Top}}(\underline{\Delta^n}, S)$$

with the images of the maps of Δ induced by the face and degeneracy maps of simplices.

In simple terms, the n-simplices of the singular complex of S are the geometric n-simplices in S.

2.5. Simplicial sets from categories

Definition 2.5.1: For a category C, its *nerve* is given by

$$N(C)_n = \mathrm{Hom}_{\mathrm{Cat}}([n], C)$$

with the images of the maps of Δ given by pre-composition.

In simple terms, the n-simplices of the nerve of C are the chains of n composable arrows in C. The morphisms are compositions and restrictions for face maps, resp. identities for degeneracies.

$$\text{Objects of } C \; \begin{array}{c} \xleftarrow{\text{target}} \\ \xrightarrow{\text{identity}} \\ \xleftarrow{\text{source}} \end{array} \; \text{Morphisms of } C \; \begin{array}{c} \xleftarrow{\text{second}} \\ \longrightarrow \\ \xleftarrow{\text{composition}} \\ \longrightarrow \\ \xleftarrow{\text{first}} \end{array} \; \begin{array}{c} \text{2 composable} \\ \text{morphisms} \end{array} \; \cdots$$

Here we see the importance of degeneracy maps - they encode the identities of a category.

2.6. Characterization of simplicial sets from topological spaces or categories

Given a simplicial set, is there a way to recognize that it is obtained as a singular complex of a space, resp. a nerve of a category? This is done by the so called horn filling properties.

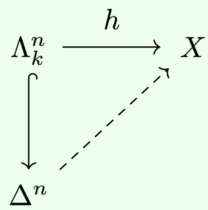
Remark: We will commit the following abuse of notation: for a map d_i and a simplicial set X, we will denote the image $X(d_i)$ just as d_i. Be aware that the direction of the map is changed! Analogously for s_i.

Definition 2.6.1: Let $\partial\Delta^n \subset \Delta^n$ be the simplicial set generated by the simplices $d_i\Delta^n$ for $i = 0, ..., n$. It is called the *boundary* of Δ^n.

Let $\Lambda^n_k \subset \Delta^n$ be the simplicial set generated by the simplices $d_i\Delta^n$ for $i = 0, ..., n, i \neq k$. It is called the *k-th horn* of Δ^n. It is an *inner* horn if $0 < k < n$ and an *outer* horn otherwise.

For a simplicial set X, a horn in X is a natural transformation $\Lambda^n_k \to X$ for some n, k.

Definition 2.6.2: We say a horn $h : \Lambda^n_k \to X$ in X admits a filler if there is a natural transformation making the following diagram commutative:



Exercise: Show that for a singular complex of a topological space, every horn admits a filler (given by projecting onto the horn).

Definition 2.6.3: A simplicial set is a *Kan complex* if every horn in it admits a filler.

Kan complexes are a good model of ∞-groupoids. It will turn out they form an ∞-category, which we will call simply *Spaces* (although lately, it's becoming popular to call it *Animae*). It will be as fundamental for ∞-categories as sets are for ordinary categories. For example, as a category has a set of morphisms between objects, an ∞-category will have a *space* of morphisms.

Regarding categories, if you haven't seen it before, the following is a great exercise to ponder:

Exercise: Show that a simplicial set is a nerve of a category if and only if every inner horn in it admits a *unique* filler (given by the unique composition in the category).

2.7. Definition of ∞-categories

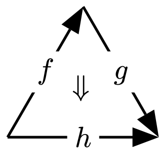
We are ready to introduce the definition that subsumes ordinary categories, as well as spaces.

Definition 2.7.1: A simplicial set is an *∞-category* if every inner horn admits a filler.

We can already start to interpret the categorical language for ∞-categories.

Definition 2.7.2: Let C be an ∞-category.

1. C_0 is the set of *objects* of C. $c \in C$ will mean $c \in C_0$.
2. C_1 is the set of *morphisms* of C. For $f \in C_1$, if $c = d_1(f)$ and $d = d_0(f)$, we will call c its *source*, d its *target* and write $f : c \to d$.
3. For an object $c \in C_0$, the identity of c is the morphism $\mathrm{id}_c = s_0(c)$.
4. If h the third boundary morphism in the filler of the horn consisting of f and g (meaning there is a 2-simplex in C_2 with the boundary as drawn bellow), we call h the *composite* of f and g and write $h \simeq f \circ g$. Note it needn't be unique!



Remark: Although the composition is not unique, we will see that the space of compositions is *contractible*. This is the right ∞-categorical analogue of uniqueness.

Definition 2.7.3: Two morphisms $f, g : c \to d$ of an ∞-category are equivalent (we write $f \simeq g$) if $f \circ \mathrm{id}_c \simeq g$ (or equivalently $g \circ \mathrm{id}_c \simeq f$).

Exercise: Show that this is an equivalence relation. Show that the different choices of composition are all equivalent.

Definition 2.7.4: A morphism $f : c \to d$ is an *isomorphism* if there is a morphism $g : d \to c$ such that $g \circ f \simeq \mathrm{id}_c$ and $f \circ g \simeq \mathrm{id}_d$.

Definition 2.7.5: An ∞-category is an ∞-groupoid if all of its morphisms are isomorphisms.

Remark: It is straightforward that every Kan complex is an ∞-groupoid. Conversely, one can prove that every ∞-groupoid is a Kan complex. This is a mathematically precise and valid version of Grothendieck's homotopy hypothesis.

2.8. Functors

One advantage of modelling ∞-categories as weak Kan complexes is that we can immediately define functors.

Definition 2.8.1: A functor F between ∞-categories C, D is a natural transformation of simplicial sets $F : C \to D$.

Exercise: Show that constructing a nerve of a category provides a fully faithful functor from 1-categories to simplicial sets.

In light of this, for ordinary 1-categories C, D, a functor between the ∞-categories $N(C)$, $N(D)$ corresponds to an ordinary functor between C and D.

Functors between two ∞-categories form an ∞-category again. To construct it, we first enrich the category of simplicial sets over itself.

Definition 2.8.2: For simplicial sets S, T, we define the simplicial set $\underline{\mathrm{Hom}}(S, T)$ (called *simplicial hom*) by

$$\underline{\mathrm{Hom}}\,(S, T)_n := \mathrm{Nat}(S \times \Delta^n, T)$$

with the images of morphisms of Δ induced in the first coordinate.

For simplicial sets S, T, U, the composition $\underline{\mathrm{Hom}}(S, T) \times \underline{\mathrm{Hom}}(T, U) \to \underline{\mathrm{Hom}}(S, U)$ is given on n simplices by

$$\mathrm{Hom}(S \times \Delta^n, T) \times \mathrm{Hom}(T \times \Delta^n, U) \xrightarrow{\circ} \mathrm{Hom}(S \times \Delta^n \times \Delta^n, U) \to \mathrm{Hom}(S \times \Delta^n, U)$$

with the last map provided by the restriction along the diagonal $\Delta^n \to \Delta^n \times \Delta^n$.

Exercise: Show that this satisfies the axioms of a category enriched over simplicial sets. Moreover, show that this is the exponential object in the category of simplicial sets (i.e $\underline{\mathrm{Hom}}(X, -)$ is the right adjoint to $X \times -$ for every simplicial set X.

Theorem 2.8.1: For an ∞-category C and a simplicial set K, $\underline{\mathrm{Hom}}(K, C)$ is an ∞-category, which we will call $\mathrm{Fun}(K, C)$. Moreover, if K and C are ∞-groupoids, $\mathrm{Fun}(K, C)$ is again an ∞-groupoid.

Notice that K doesn't have to be an ∞-category (and it will sometimes be advantageous for it not to be so, so that we don't have to specify all the higher morphisms). We will not prove this theorem, as it would involve too long of a detour into the combinatorics of simplicial sets, but we will sketch the basic ideas. By adjunction, we want to show lifts of the problems of the following form:

$$\begin{array}{ccc} \Lambda^n_k \times K & \longrightarrow & C \\ \big\downarrow & \nearrow & \\ \Delta^n \times K & & \end{array}$$

We can turn this into a more symmetric problem by considering the unique map $C \to \Delta^0$.

$$\begin{array}{ccc} \Lambda^n_k \times K & \longrightarrow & C \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n \times K & \longrightarrow & \Delta^0 \end{array}$$

Definition 2.8.3: A map of simplicial sets $f : X \to Y$ is called an *(inner) fibration* if it has *right lifting property* for every (inner) horn inclusion, meaning there exists a lift in every diagram of the following form:

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \Delta^n & \longrightarrow & Y \end{array}$$

Dually, a map g is called *(inner) anodyne* if it has *left lifting property* with respect to (inner) fibrations (a similar diagram with g on the right and an (inner) fibration on the left).

The proof now proceeds by showing that the class of inner anodyne maps is closed under $- \times K$.

Definition 2.8.4: A functor between ∞-categories $f : C \to D$ is an *equivalence* if there is a functor $g : D \to C$, along with natural isomorphisms $f \circ g \simeq \mathrm{id}$, $g \circ f \simeq \mathrm{id}$.

Remark: For ordinary categories, this corresponds to ordinary equivalence of categories; for Kan complexes, to weak homotopy equivalence.

2.9. Mapping spaces

Definition 2.9.1: For objects c, d in an ∞-category C, the mapping space $\mathrm{Map}(a, b)$ is the following pullback (of simplicial sets):

$$\begin{array}{ccc} \mathrm{Map}(c, d) & \longrightarrow & \mathrm{Fun}(\Delta^1, C) \\ \big\downarrow & & \big\downarrow \\ \Delta^0 & \xrightarrow{(c, d)} & \mathrm{Fun}(\partial\Delta^1, C) \end{array}$$

In particular, the objects of $\mathrm{Map}(c, d)$ are morphisms between c and d and the morphisms are commutative squares where the horizontal arrows are identities.

Proposition 2.9.1: The mapping spaces are ∞-groupoids.

We are again not going to give the full proof due to its technicality. One can show that show that the map on the right is not only a fibration, but what is called *conservative* fibration, which is a class stable under pullbacks, making the map on the left a fibration.

Definition 2.9.2: Let $f : C \to D$ be a functor between ∞-categories. We say that f is *fully faithful* if for every $c, d \in C$, the induced map of mapping spaces

$$\mathrm{Map}_C(c, d) \to \mathrm{Map}_D(f(c), f(d))$$

is a homotopy equivalence.

We say f is *essentially surjective* if for every $d \in D$, there is $c \in C$ such that $f(c) \simeq d$.

Theorem 2.9.1: A functor $f : C \to D$ is an equivalence of ∞-categories iff it is fully faithful and essentially surjective.

The proof again involves some combinatorics with fibrations.

Remark: Similarly, we may consider the pullback

$$\begin{array}{ccc} \mathrm{Map}(c, d, e) & \longrightarrow & \mathrm{Fun}(\Delta^2, C) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{(c,d,e)} & C \times C \times C \end{array}$$

It turns out the restriction map $\mathrm{Map}(c, d, e) \overset{d_0, d_2}{\to} \mathrm{Map}(c, d) \times \mathrm{Map}(d, e)$ is a homotopy equivalence. Composition of morphisms now amounts to *choosing* a homotopy inverse

$$\mathrm{Map}(c, d) \times \mathrm{Map}(d, e) \to \mathrm{Map}(c, d, e) \overset{d_1}{\to} \mathrm{Map}(c, e)$$

2.10. ∞-categories from simplicially enriched categories

It would be convenient to have at our disposal the whole ∞-category of spaces, or of (small) ∞-categories. In the sections above, we saw how to present them as categories with simplicial sets of morphisms. We will now develop a construction to turn them into ∞ categories in our sense (i.e. weak Kan complexes). Here, will denote the category of simplicial categories by SCat.

Definition 2.10.1: For two natural numbers $i \leq j$, denote as $P_{i,j}$ the poset of subsets $\{I \subset \{i, ..., j\} \mid i \in I \wedge j \in I\}$ ordered by inclusion.

Let $\mathfrak{C}[\Delta^n]$ be the simplicially enriched category with

- objects $0, ..., n$
- simplical hom-sets $\underline{\mathrm{Hom}}(i, j) = P_{i,j}$
- composition induced by union of subsets

$\mathfrak{C}[\Delta^n]$ is to be considered as a "thickened" version of $[n]$, where the simplicial set of maps between two elements is again empty or contractible, but now forming a simplicial cube $N(P_{i,j})$.

Definition 2.10.2: For a simplicial category S, let its *homotopy coherent nerve* be given by

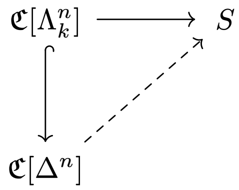
$$N(S)_n := \mathrm{Hom}_{\mathrm{SCat}}(\mathfrak{C}[\Delta^n], S)$$

with the images of maps of Δ given by pre-composition.

Exercise: Show that $\mathfrak{C}$ can be extended to a functor from simplicial sets to simplicial categories, which is the left adjoint to the homotopy coherent nerve functor (by a similar formula to the geometric realization - the values are given on standard simplices Δ^n and left Kan extended from there).

Theorem 2.10.1: Let S be a simplicially enriched category such that for all objects $s, t \in S$, the simplicial set $\underline{\text{Hom}}(s, t)$ is a Kan complex. Then the homotopy coherent nerve $N(S)$ is an ∞-category.

Proof: By adjunction, we have to solve the following lifting problem in simplicial categories



The simplicial category $\mathfrak{C}[\Lambda^n_k]$ has the same objects and simplicial homs as $\mathfrak{C}[\Delta^n]$, except between the objcets 0 and n, where it is the nerve of poset $P_{0,n}$ without the maximal set and the set of all elements except k. Extending it to $N\left(P_{0,n}\right)$ is possible because the simplicial homs in S are Kan complexes. □

Definition 2.10.3: Let Spc be the homotopy coherent nerve of the simplicial category of Kan complexes. We will call this infinity category *spaces*.

Let qCat be the simplicial enriched category, whose objects are small ∞-categories and the simplicial hom between ∞-categories C, D be the maximal Kan complex in $\text{Fun}(C, D)$. Let $\infty - \text{cat}$ be the homotopy coherent nerve of qCat.

Remark: For Kan complexes S, T, it can be shown that $\underline{\text{Hom}}(S, T) \simeq \text{Map}_{\text{Spc}}(S, T)$.