Definition 1. Let C, D be ∞-categories. An adjunction between C and D is a pair of functors $f : C \to D$ and $g : D \to C$ together with a unit $\eta : 1_C \to gf$ and counit $\varepsilon : fg \to 1_D$ natural transformations such that there exist 2-simplices

$$\begin{array}{ccc} f & \xrightarrow{\eta} & fgf \\ & \searrow^{id} & \downarrow{\varepsilon} \\ & & f \end{array} \quad \text{in } \mathrm{Fun}(C, D) \qquad \begin{array}{ccc} g & \xrightarrow{\eta} & gfg \\ & \searrow^{id} & \downarrow{\varepsilon} \\ & & g \end{array} \quad \text{in } \mathrm{Fun}(D, C)$$

We write $f \dashv g$ and say that f is left adjoint, g is right adjoint.

Exercise 2. Show that an adjunction gives rise to a homotopy equivalence of spaces

$$\mathrm{Map}_D(f(c), d) \xrightarrow{g} \mathrm{Map}_C(gf(c), g(d)) \xrightarrow{\eta^*} \mathrm{Map}_C(c, g(d)) \tag{1}$$

Definition 3 (Long awaited simplicial homotopy...)**.** A homotopy between two morphisms of simplicial objects $f, g : X \to Y$ is a morphism $H : X \times \Delta^1$ that makes the following diagram commute

$$\begin{array}{ccccccccc} X & \cong & X \times \Delta^0 & \xrightarrow{id \times \delta^1} & X \times \Delta^1 & \xleftarrow{id \times \delta^0} & X \times \Delta^0 & \cong & X \\ & & & \searrow^{f} & \downarrow{H} & \swarrow^{g} & & & \\ & & & & Y & & & & \end{array}$$

or equivalently, if for each p there exist functions $h_i = h_i^p : X_i \to Y_{i+1}$, for each $0 \leq i \leq p$, such that

1.
$$d_0 h_0 = f$$
$$d_{p+1} h_p = g$$

2.
$$d_i h_j = h_{j-1} d_i \quad \text{if } i < j$$
$$d_{j+1} h_{j+1} = d_{j+1} h_j$$
$$d_i h_j = h_j d_{i-1} \quad \text{if } i > j+1$$

3.
$$s_i h_j = hj + 1 s_i \quad \text{if } i \leq j$$
$$s_i h_j = h_j si - 1 \quad \text{if } i > j.$$

Remark 4. The converse of the previous exercise is also true but harder to prove. In fact, even less data is needed to produce an adjunction.

More precisely, suppose we are given a functor $g : D \to C$. To produce an adjunction $f \dashv g$,it is enough to give, for each $c \in C$ an object $f(c) \in C$, and a morphism $c \to g(f(c))$ such that the condition (1) is an equivalence for all $d \in D$.

More formally, the claim is that there is a functor $f : C \to D$ sending c to $f(c)$ such that $f \dashv g$. Moreover, the unit natural transformation $\eta : id_C \to gf$ of the adjunction can be chosen homotopic to the given maps.

1 $\mathbf{Spc} \hookrightarrow \mathbf{Cat}_\infty$

Given an ∞-category C, we can produce its maximal ∞-subgrupoid $C^{\simeq}$, this assembles into a functor $(-)^{\simeq} : qCat \to Kan$, moreover

$$Kan \underset{(-)^{\simeq}}{\overset{inc}{\rightleftarrows}} qCat \quad (\perp)$$

where the unit is an identity and counit is an inclusion of a subgrupoid.

The homotopy coherent nerve preserves the triangle identities, hence

$$\mathbf{Spc} \underset{(-)^{\simeq}}{\overset{inc}{\rightleftarrows}} \mathrm{Cat}_\infty \quad (\perp)$$

On the other hand, given an ∞-category C, there is a functorial way to add inverses to all morphisms. Let J be a category with two objects and one isomorphism between them, view it as an ∞-category through the nerve. Consider a pushout

$$\begin{array}{ccc} \coprod_{C_1} \Delta^1 & \hookrightarrow & C \\ \downarrow & sSet & \downarrow \\ \coprod_{C_1} J & \longrightarrow & D \end{array}$$

The resulting D may not be an ∞-category, but we can make it into one by adding fillers $D \to \widetilde{D}$ in countably many steps. This assignment extends to a functor and

$$\begin{array}{ccc} \mathrm{Map}_{\mathbf{Spc}}(\widetilde{D}, X) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathrm{Cat}_\infty}(C, X) \\ & \searrow^{\simeq} \quad \nearrow^{\simeq} & \\ & \mathrm{Map}_{\mathrm{Cat}_\infty}(\widetilde{D}, X) & \end{array}$$

2 Propositions

Proposition 5. For an ∞-category C TFAE:

(i) C admits I-shaped colimits;

(ii) $\underline{(-)} : C \to C^I$ has a left adjoint $colim_I : C^I \to C$.

Proposition 6. Let I, K be simplicial sets and C an ∞-category that admits I-shaped (co)limits. Then $\mathrm{Fun}(K, C)$ also admits I-shaped (co)limits and these are computed pointwise. That is, the family of functors, for $k \in K$,

$$ev_k : \mathrm{Fun}(K, C)$$

preserves and reflects I-shaped (co)limits.

Proposition 7. Let $f : C \to D$ be a left adjoint, then f preserves all colimits.