Reading seminar on ∞-categories

1. Colimits

Definition 1.1: Let $\mathsf{flip} : \Delta \to \Delta$ be the functor constant on objects and mapping d_i to d_{n-i} in dimension n, analogously for the degeneracy map.

For a simplicial set $S : \Delta^{\mathrm{op}} \to \mathsf{Set}$, the opposite simplicial set is the composite $S \circ \mathsf{flip}$.

Using this, we can cheat in defining colimits if we know the definition of limits.

Definition 1.2: For an ∞-category C, a colimit in C is a limit in C^{op}.

Explicitly, a cone under $F : I \to C$ is a pair (ℓ, η) where $\ell \in C$ and $\eta : F \to \underline{\ell}$ is a natural transformation. We then require the following homotopy equivalence for each $c \in C$

$$\mathsf{Map}_C(\ell, c) \to \mathsf{Map}_{C^I}(F, \underline{c})$$

Example: $\ell \in C$ is *initial* if $\mathsf{Map}_C(\ell, c)$ is contractible for each $c \in C$. The coproduct is characterized my $\mathsf{Map}_C(\sqcup_{i \in I} F(i), c) \simeq \sqcap_{i \in I} \mathsf{Map}_C(F(i), c)$.

Definition 1.3: We say a simplicial set is *finite* if it has finitely many non-degenerate simplices.

Theorem 1.1: If C admits pushouts and finite coproducts, then C admits all finite colimits.

Proof sketch: Suppose we have a functor $F : I \to C$. Consider the skeletal filtration

$$I_0 \subseteq \ldots \subseteq I_n = I$$

with the simplicial set I_k containing non-degenerate simplices of dimension at most k. We do induction on n. $n = 0$ amounts to coproducts. For the inductive step, we have the following pushout (of attaching n-cells) in the simplicial sets

$$\begin{array}{ccc} \sqcup\, \partial\Delta_n & \longrightarrow & I_{n-1} \\ \downarrow & & \downarrow \\ \sqcup\, \Delta_n & \longrightarrow & I_n \end{array}$$

By the inductive assumption, F restricted to the objects in the top row has a colimit in C. The colimits over Δ_n always exist since it has a terminal object. It turs out that in this case, the pushout of these 3 colimits is actually the colimit of F. □

Remark: Generalizing this argument, one can show that if C admits pushouts and all coproducts, then it admits all colimits.

2. Alternative definitions

Here we do just a brief survey of alternative definitions of (co)limits found in the literature.

Definition 2.1: For simplicial sets S, T, define their join $S \star T$ by

$$(S \star T)_n := S_n \sqcup \bigsqcup_{i+j=n+1} S_i \times T_j \sqcup T_n$$

where the components in the disjoint union correspond to cuts of the linearly ordered set $0 < \ldots < n$ (with S_n, resp. T_n corresponding to the cut below, resp. above). The images of morphisms of Δ are induced by their maps on cuts.

For a simplicial set I, the simplicial set $I^{\triangleleft} = \Delta_0 \star I$ is called the *left cone* on I and the simplicial set $I^{\triangleright} := I \star \Delta_0$ is called the *right cone* on I.

Definition 2.2: For $F : I \to C$ a map of simplicial sets, define the slice over F, denoted $C_{/F}$ as the simplicial set with the universal property for each $K \in \mathsf{SSet}$:

$$\mathsf{Hom}_{\mathrm{SSet}}\left(K, C_{/F}\right) = \mathsf{Hom}_F(K \star I, C)$$

where on the right hand side, we take only the maps which restrict to F on I.

Similarly, define the slice under F, denoted $C_{F/}$ via the universal property for each $K \in \mathrm{SSet}$:

$$\mathsf{Hom}_{\mathrm{SSet}}\left(K, C_{F/}\right) = \mathsf{Hom}_F(I \star K, C)$$

Remark: If C is an ∞-category, then so are $C_{/F}$ and $C_{F/}$.

Proposition 2.1: From a cone (ℓ, η) (our previous definition) over a functor $F : I \to C$, we can define a functor $\tilde{F} : I^{\triangleleft} \to C$ and an element $\tilde{\eta} \in C_{/F}$ such that the following are equivalent:

- (ℓ, η) is a limit cone over F
- restriction $I \hookrightarrow I^{\triangleleft}$ induces a homotopy equivalence for every $c \in C$

$$\mathsf{Map}_{C^{I^{\triangleleft}}}\left(\underline{c}, \tilde{F}\right) \to \mathsf{Map}_{C^{I}}(\underline{c}, F)$$

- $\tilde{\eta}$ is the terminal object of $C_{/F}$

3. Adjunctions

Definition 3.1: For C, D ∞-categories, and adjunction is a pair of functors $f : C \to D$, $g : D \to C$, along with the unit $\eta : \mathrm{id}_C \to gf$ and the counit $\varepsilon : fg \to \mathrm{id}_D$ transformations satisfying the triangle identities, meaning there are the following simplices in $\mathsf{Fun}(C, D)$, resp. $\mathsf{Fun}(D, C)$: