Convention: I - simplicial set, C - ∞-category, $F : I \to C$ a simplicial map. Given an object $l \in C$, we denote by

$$\underline{l} : I \to \Delta^0 \xrightarrow{l} C$$

the constant functor with value l.

1 Recall:

We have defined an ∞-*category of functors* $\mathrm{Fun}(I, C)$ (or C^I) by

$$\mathrm{Fun}(I, C)_n = \mathrm{Hom}_{sSet}(I \times \Delta^n, C).$$

The *mapping space* $\mathrm{Map}_C(c, d)$ between $c, d \in C$ in the pullback

$$\begin{array}{ccc} \mathrm{Map}_C(c,d) & \longrightarrow & \mathrm{Fun}(\Delta^1, C) \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & C \times C \end{array}$$

In ordinary category theory, a for a functor $F : K \to C$, $\lim F$ is a *universal cone*, or in other words it is a pair (l, η), where $l \in C$ and $\eta : \underline{l} \to F$ such that there exists a natural bijection, for each $c \in C$

$$\mathrm{Hom}(c, \lim F) \cong \mathrm{Nat}(\underline{c}, F).$$

2 Limits

Definition 1. A *cone* over F is a pair (l, η), where $l \in C$ and $\eta \in \mathrm{Fun}(I, C)_1$ is a natural transformation $\eta : \underline{l} \to F$. It is a *limit* cone, if for each $c \in C$ the following is a homotopy equivalence

$$\mathrm{Map}_C(c, l) \xrightarrow{\underline{(-)}} \mathrm{Map}_{C^I}(\underline{c}, \underline{l}) \xrightarrow{\eta_*} \mathrm{Map}_{C^I}(\underline{c}, F)$$

In the definition above, the map $\mathrm{Map}_C(c, l) \xrightarrow{\underline{(-)}} \mathrm{Map}_{C^I}(\underline{c}, \underline{l})$ is induced by $\underline{(-)} : C \to \mathrm{Fun}(I, C)$. The map $\mathrm{Map}_{C^I}(\underline{c}, \underline{l}) \xrightarrow{\eta_*} \mathrm{Map}_{C^I}(\underline{c}, F)$ is postcomposition with η.

Exercise 2. Show that any two limits are isomorphic.

Exercise 3. Check out that in case I, C are (nerves of) ordinary categories, then Definition 1 recovers the ordinary notion of limits.

Example 4. Suppose I is a discrete ∞-category. Then $C^I \cong \prod_I C$. So

$$\mathrm{Map}_{C^I}(\underline{c}, F) \cong \mathrm{Map}_{\prod_I C}(\underline{c}, F) \cong \prod_i \mathrm{Map}_C(c, F(i)).$$

Then $\mathrm{Map}_C(c, \lim F) \xrightarrow{\sim} \prod_i \mathrm{Map}_C(c, F(i))$. Therefore a morphism into product is up to homotopy equivalence a family of morphisms into $F(i)$, similar to ordinary categories

Exercise 5. Put $I = \emptyset$. What is $\mathrm{Fun}(I, C)$? Apply the definition of limit and characterize terminal objects in C.

Exercise 6 (Pullbacks)**.** Put $I = \Lambda^2_2$, then $F : I \to C$ is given by a following diagram in C.

$$\begin{array}{ccc} & & r \\ & & \downarrow{\scriptstyle f} \\ s & \xrightarrow[g]{} & t \end{array}$$

Sketch that a map into the pullback (if exists) $r \times_t s$ is given by a commutative square (what is a 'commutative square' in ∞-categories?)

$$\begin{array}{ccc} c & \xrightarrow{h} & r \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ s & \xrightarrow[g]{} & t \end{array}$$

where $fh \simeq gi$. Notice how the strict equality in ordinary categories is replaced by the equivalence.

Proposition 7. $\mathbf{Spc}$, $\infty - \mathbf{cat}$ have all small limits. The inclusion $\mathbf{Spc} \hookrightarrow \infty - \mathbf{cat}$ preserves all small limits.

2.1 Limits in Spc

We state a following lemma without proof and derive several results about limits in the ∞-category **Spc**.

Lemma 8. Let C be a Kan-enriched category, $N_\Delta(C)$ it's homotopy coherent nerve, $F : I \to N_\Delta(C)$ a functor, $z \in \mathbf{Spc}$, $x \in C$. Then

$$\underline{\mathrm{Hom}}_{Kan}(z, \mathrm{Map}_{C^I}(\underline{x}, F)) \simeq \mathrm{Map}_{\mathbf{Spc}^I}(\underline{z}, F(-)).$$

In case $C = \mathbf{Spc}$, these are also equivalent to $\mathrm{Map}_{\mathbf{Spc}^I}(\underline{z \times x}, F)$.

Proposition 9. A cone (y, η) over F in $\mathbf{Spc}$ is a limit cone iff for each $x \in \mathbf{Spc}$, the map

$$\pi_0(\mathrm{Map}_{\mathbf{Spc}}(x, y) \to \pi_0(\mathrm{Map}_{\mathbf{Spc}^I}(\underline{x}, F)$$

is an isomorphism.

Proof. The left to right implication is immediate. In the other direction, suppose for each $x \in \mathbf{Spc}$, the said map is an isomorphism. This is the same as to say that the set of equivalence classes of morphisms from x to y under the homotopy equivalence relation $[x, y]_{\mathbf{Spc}}$ is in natural bijection with $[\underline{x}, F]_{\mathbf{Spc}^I}$.

We prove that $[z, \mathrm{Map}_{\mathbf{Spc}}(x, y)]_{\mathbf{Spc}} \cong [z, \mathrm{Map}_{\mathbf{Spc}^I}(\underline{x}, F)]_{\mathbf{Spc}}$ for any $z \in \mathbf{Spc}$. By Yoneda lemma, this implies $\mathrm{Map}_{\mathbf{Spc}}(x, y) \simeq \mathrm{Map}_{\mathbf{Spc}^I}(\underline{x}, F)]_{\mathbf{Spc}}$.

Let $z \in \mathbf{Spc}$. Then the following diagram commutes.

$$\begin{array}{ccc} [z, \mathrm{Map}_{\mathbf{Spc}}(x, y)]_{\mathbf{Spc}} & \xrightarrow{(1)} & [z \times x, y]_{\mathbf{Spc}} \\ {\scriptstyle (3)}\downarrow & & \downarrow{\scriptstyle (2)} \\ [z, \mathrm{Map}_{\mathbf{Spc}^I}(\underline{x}, F)]_{\mathbf{Spc}} & \xrightarrow[(4)]{} & [\underline{z \times x}, F]_{\mathbf{Spc}^I} \end{array}$$

We comment on the maps above

(1) is the application of the observation $\mathrm{Map}_{\mathbf{Spc}}(x, y)]_{\mathbf{Spc}} = \underline{\mathrm{Hom}}_{sSet}(x, y)$ and of simplicial inner hom adjunction;

(2) is given by assumption;

(3) is given by post-composition with the limit property;

(4) is given by Lemma 8.

Moreover, (1), (2), (3) are isomorphisms, so (3) has to be one.

■

Proposition 10. For any functor $F : I \to \mathbf{Spc}$, the space $\mathrm{Map}_{\mathbf{Spc}^I}(\underline{\Delta^0}, F)$ is the limit of F. Therefore $\mathbf{Spc}$ has all small limits.

Sketch. For each $x \in \mathbf{Spc}$,

$$\mathrm{Map}_{\mathbf{Spc}}(x, \mathrm{Map}_{\mathbf{Spc}^I}(\underline{\Delta^0}, F)) \xrightarrow[\text{Lemma 8}]{\sim} \mathrm{Map}_{\mathbf{Spc}^I}(\underline{x \times \Delta^0}, F) \xrightarrow{\sim} \mathrm{Map}_{\mathbf{Spc}^I}(\underline{x}, F)$$

The cone is given by putting $x = \mathrm{Map}_{\mathbf{Spc}^I}(\underline{\Delta^0}, F)$ and computing the image of the identity. ■

Exercise 11. Let $x \in \mathbf{Spc}$ and $F : I \in \mathbf{Spc}$ a constant functor on x ($\underline{x}$ in our notation). Compute the $\lim F$.

3 Colimits

Definition 12 (Dual)**.** A *cone* under F is a pair (l, η), where $l \in C$ and $\eta \in \mathrm{Fun}(I, C)_1$ is a natural transformation $\eta : F \to \underline{l}$. It is a *colimit* cone, if for each $c \in C$ the induced map

$$\mathrm{Map}_C(l, c) \xrightarrow{\sim} \mathrm{Map}_{C^I}(F, \underline{c})$$

is a homotopy equivalence.

The following examples are similar to the case of limits.

Example 13. (i) $l \in C$ is initial iff $\mathrm{Map}(l, c)$ is contractible for all $c \in C$.

(i) $\mathrm{Map}_C(\coprod_i F(i), c) \cong \prod_i \mathrm{Map}_C(F(i), c)$.

Example 14. For pushouts there is a formula

$$\mathrm{Map}_C(y \coprod_x z) \simeq \mathrm{Map}_C(y, c) \times_{\mathrm{Map}_C(x,c)} \mathrm{Map}_C(z, c),$$

where the right-hand side denotes the pullback in $\mathbf{Spc}$.

Proposition 15. TFAE

(i) C admits (finite) colimits

(ii) C admits coequalizers and (finite) coproducts.

(iii) C admits pushouts and (finite) coproducts.

The dual statement hold for limits.