Fundamental group: homework

Send the solutions to *maros@grego.site* or hand them in person.

You may use without proof the following fact:
A (hollow) torus is homeomorhic to the product $\mathbb{S}^1 \times \mathbb{S}^1$ (which may be considered a subset of $\mathbb{R}^4$).

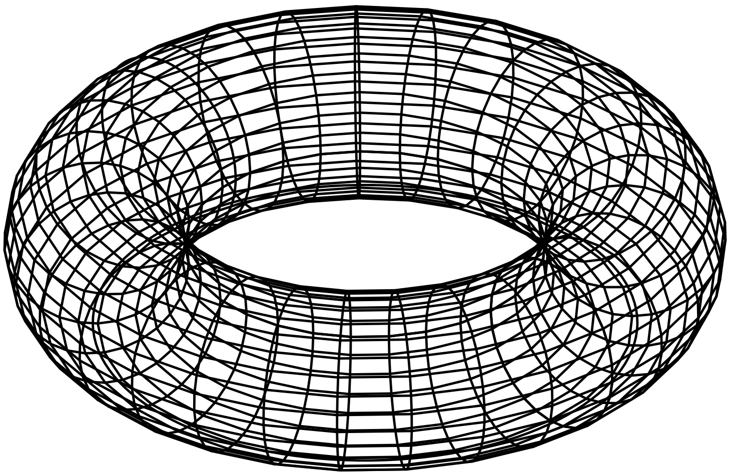
Let p be the covering $\mathbb{R}^1 \to \mathbb{S}^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$. Consider the map $\mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ given by $(s, t) \mapsto (p(s), p(t))$.

1. Show that it is a covering. (1 point)

2. Describe the fiber (preimage) of the point $(1, 0) \times (1, 0) \in \mathbb{S}^1 \times \mathbb{S}^1$. (1 point)

3. Compute the group of its deck transformations and deduce from it the fundamental group of the torus. (2 points)

4. (bonus) Describe some loops on the torus that generate its fundamental group. Use the lifts of these loops to the universal cover to show graphically that the fundamental group of the torus is commutative. (2 points)



If a set G has a group structure with a multiplication $*$ (which we will denote $(G, *)$) and a unit e, the set $G \times G$ has naturally a group structure with multiplication $(g_1, h_1) * (g_2, h_2) = (g_1 * g_2, h_1 * h_2)$ for $g_1, g_2, h_1, h_2 \in G$ and the unit (e, e) (which we will denote $(G \times G, * \times *)$)

5. (Eckmann-Hilton argument) Consider a set G with two group structures given by operations $*$ and $\odot$ which are homomorphisms for each other, i.e.:

- $* : G \times G \to G$ is a homomorphism $(G \times G, \odot \times \odot) \to (G, \odot)$,
- $\odot : G \times G \to G$ is a homomorphism $(G \times G, * \times *) \to (G, *)$.

Using the definition of group homomorphism, show that $*$ and $\odot$ are the same operation and moreover that it it commutative. (2 points)

Note: This shows that higher homotopy groups $\pi_i(S)$ for $i \geq 2$ are commutative for every space S, since we can define "horizontal" and "vertical" composition for them.