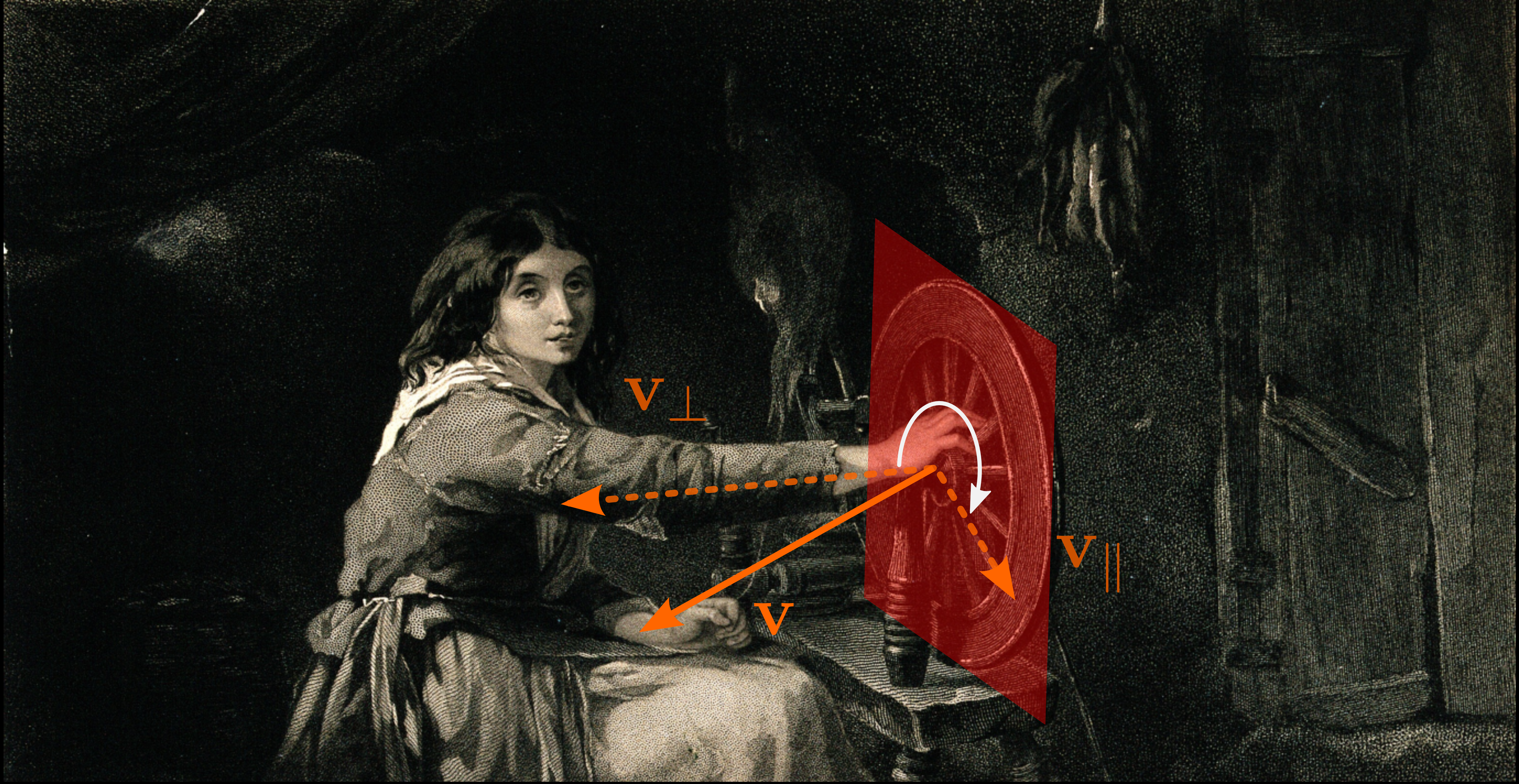
n-dimensional rotations using geometric (Clifford) algebra

Maroš Grego

What is a rotation?

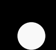
$$?$$

$$\begin{pmatrix} 0.67085835 & 0.46067794 & -0.58114104 \\ -0.16046196 & 0.85525532 & 0.49273756 \\ -0.72401729 & 0.23730607 & -0.64767646 \end{pmatrix}$$



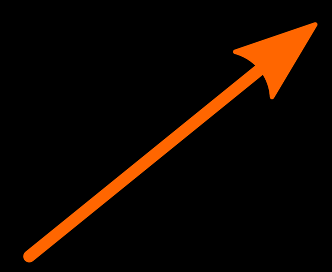
The rotation "happens" in a plane.
Only the part of $\mathbf{v}$ parallel to this plane gets rotated.

How to define the rotation plane?



0D

scalar

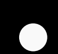


1D

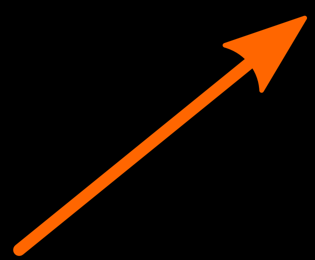
vector

oriented line segment

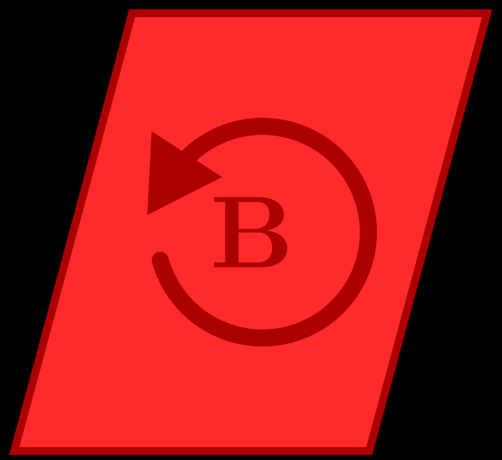
How to define the rotation plane?



0D
scalar

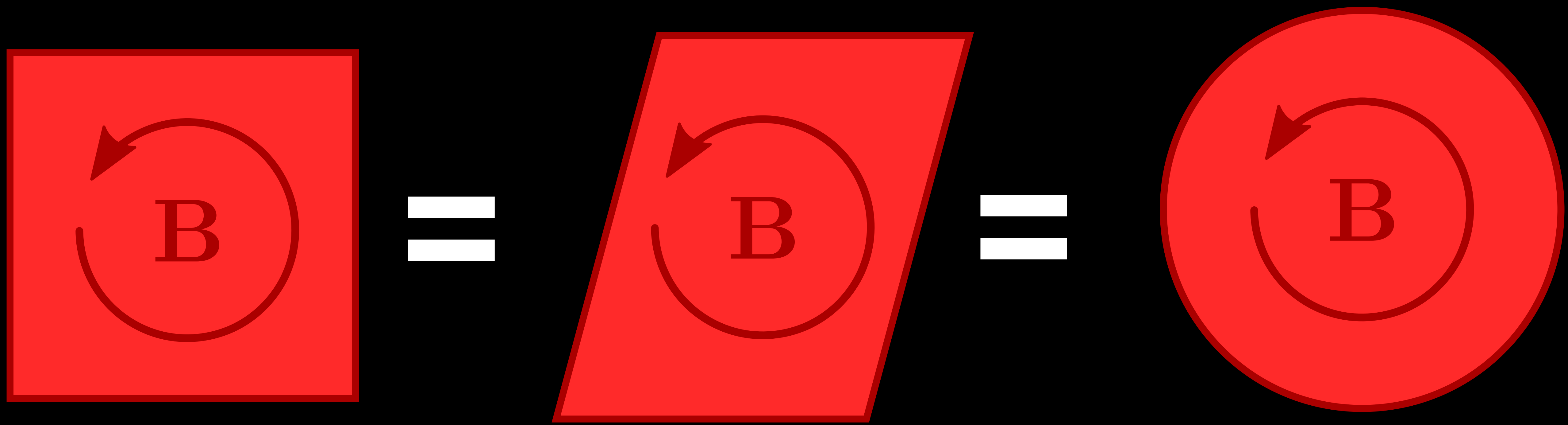


1D
vector
oriented line segment



2D
bivector
oriented area element

Only the **orientation** and **area** matter



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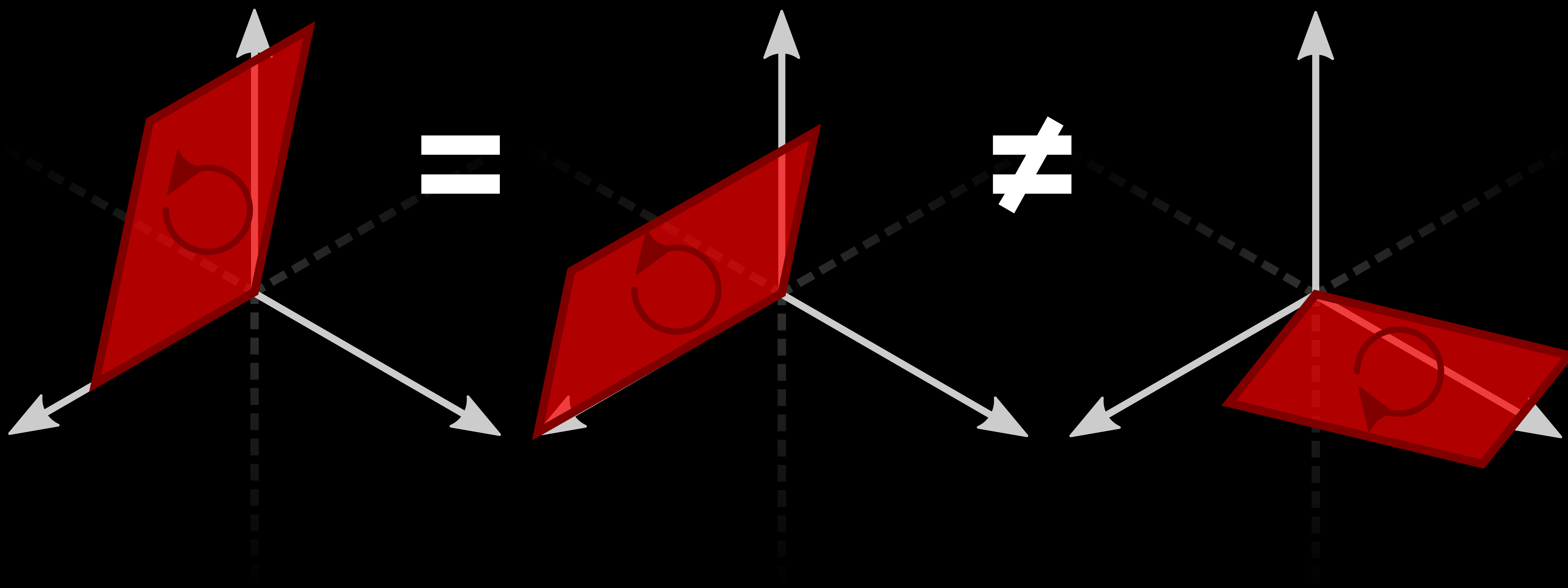
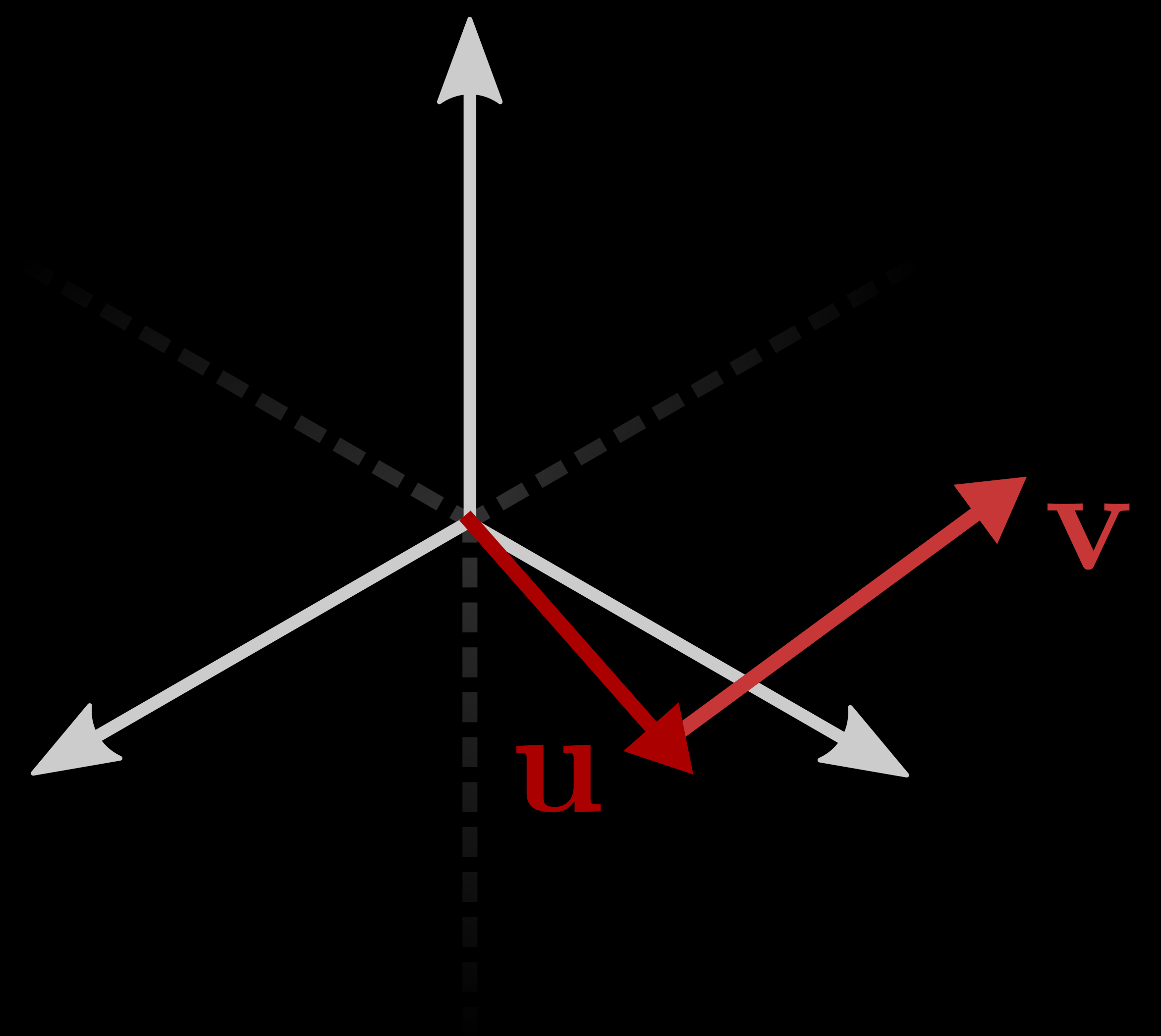


A bivector is formed from two vectors via the **exterior product**



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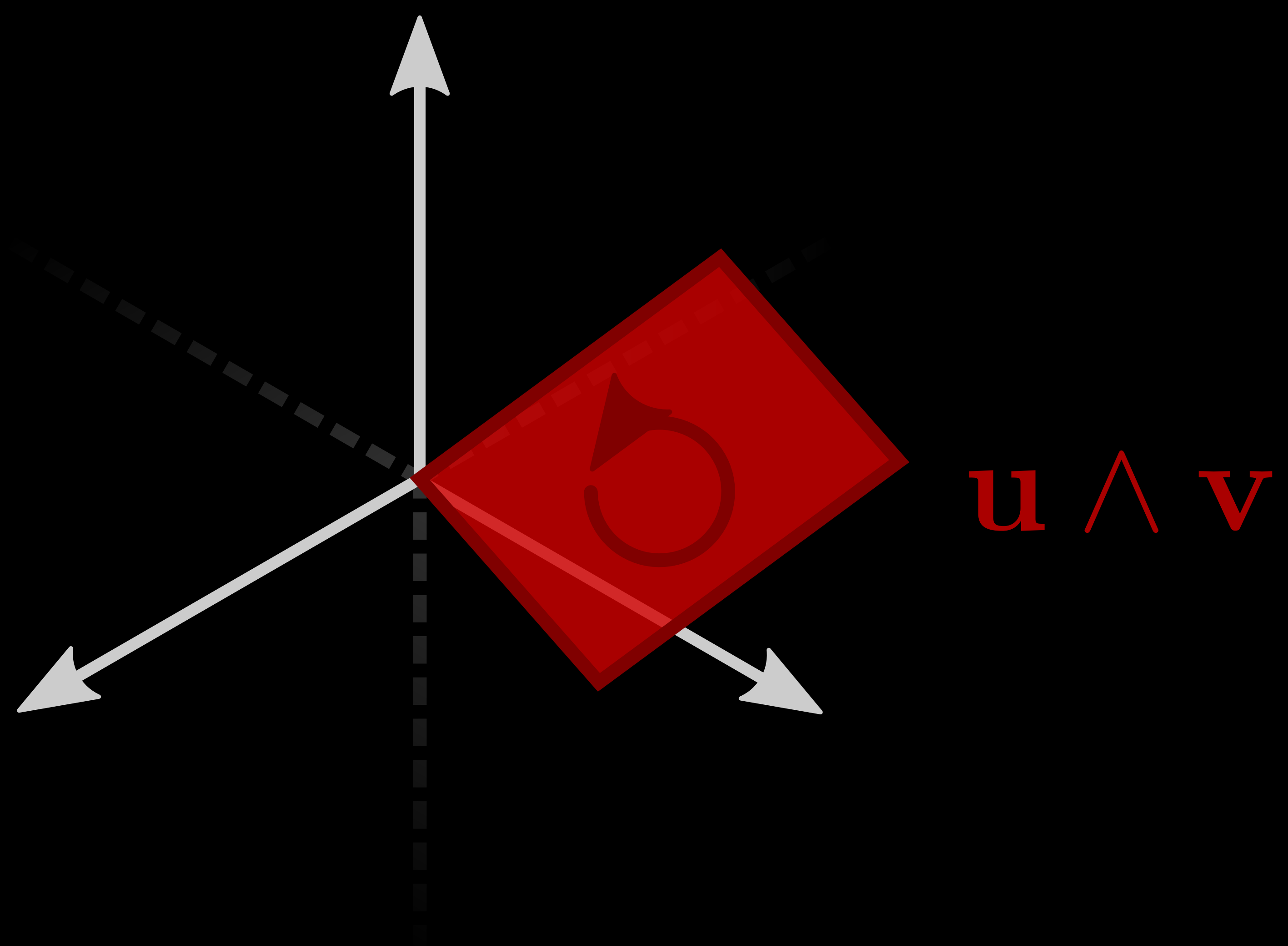
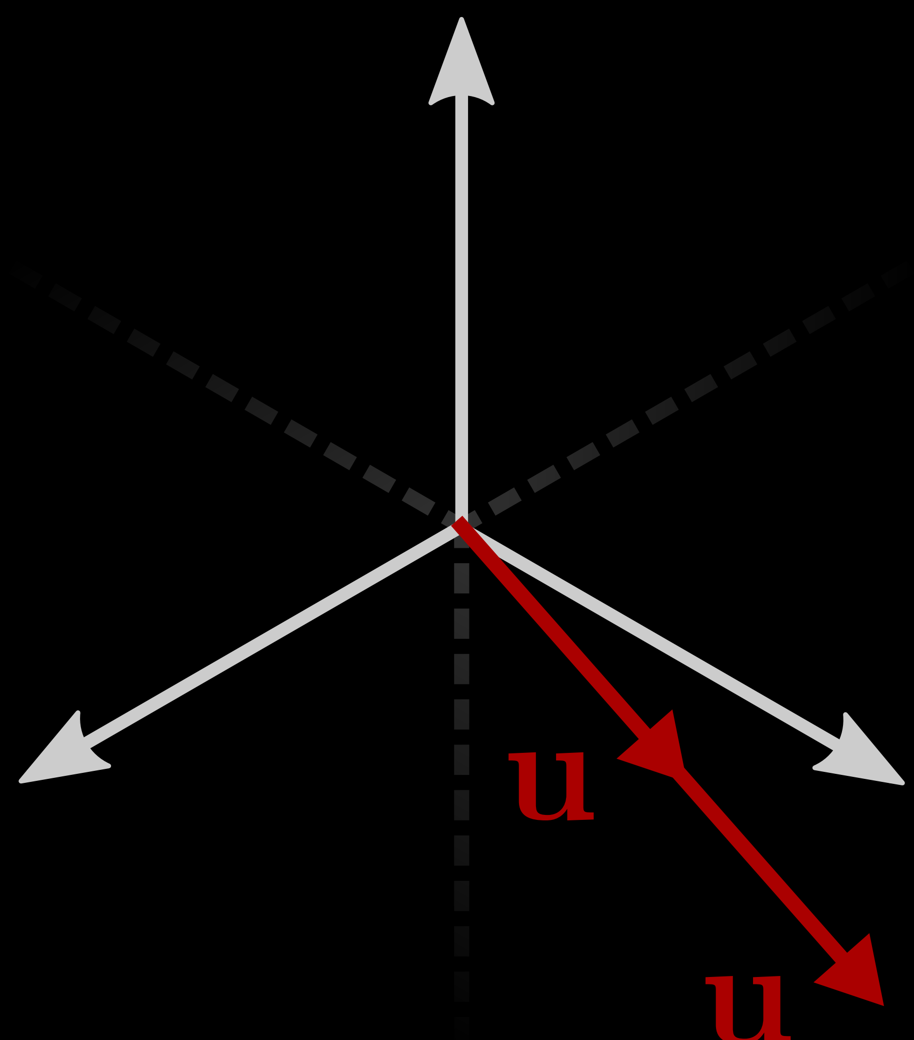
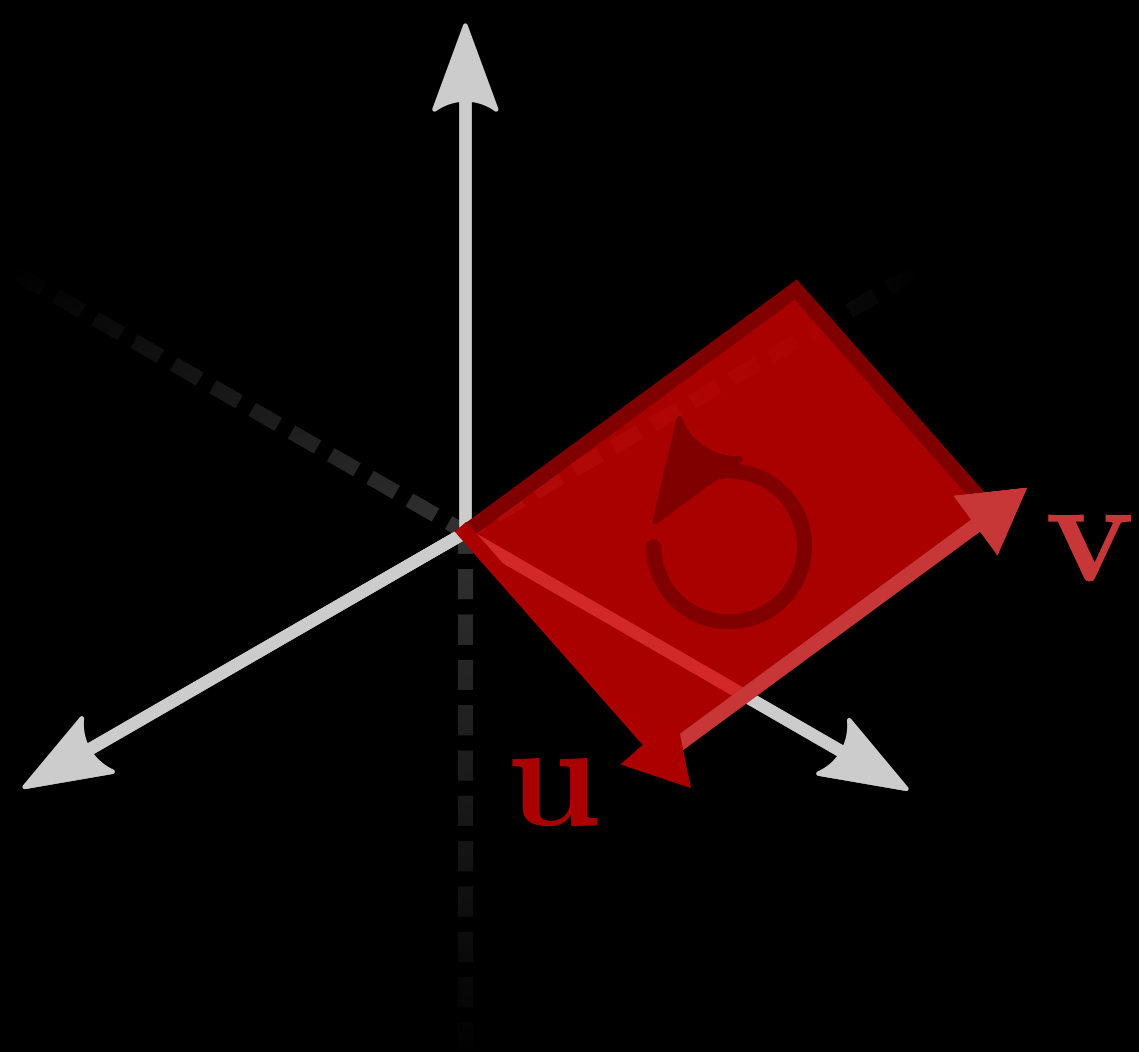
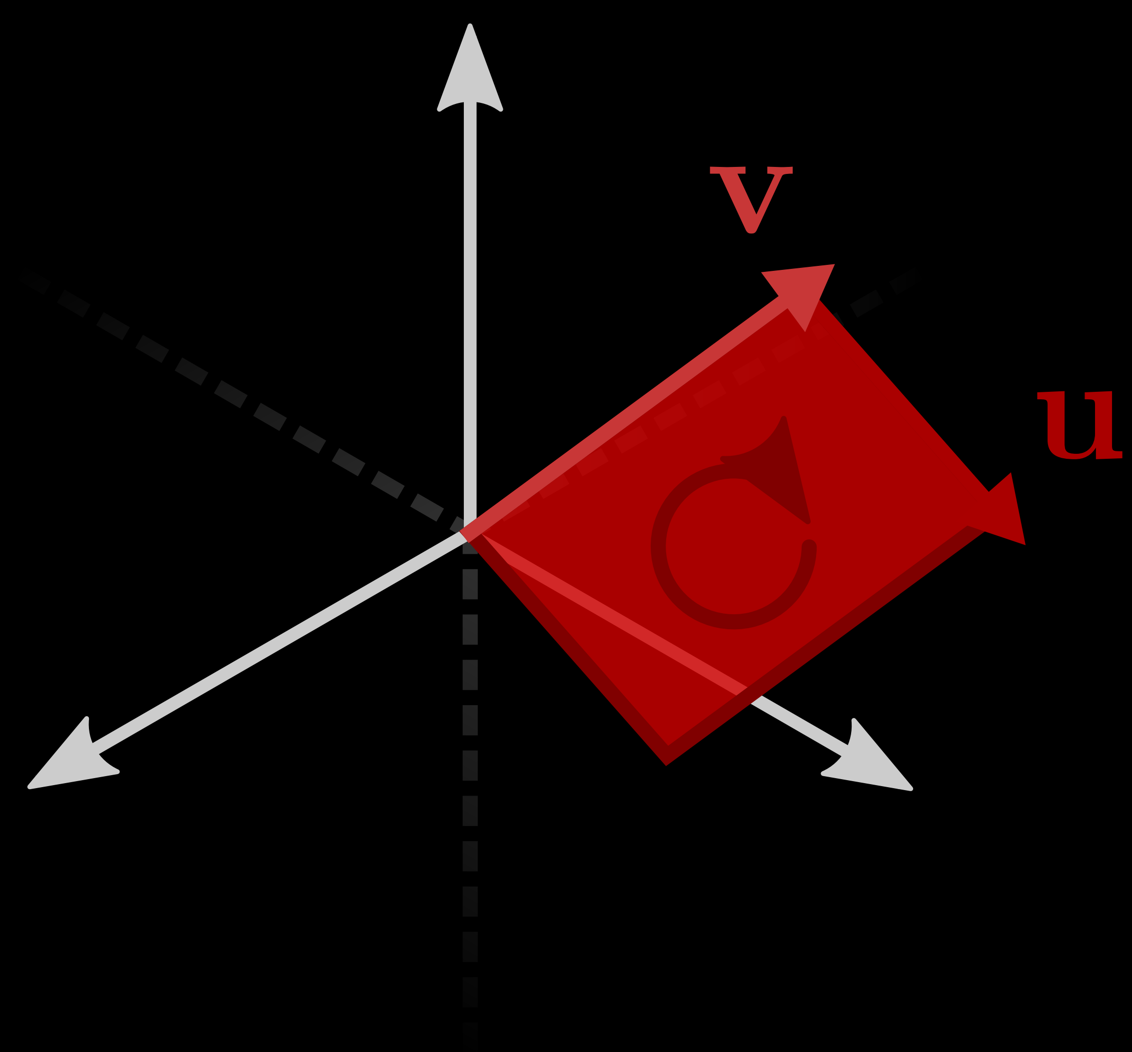


The outer product of a vector with itself has **zero area**

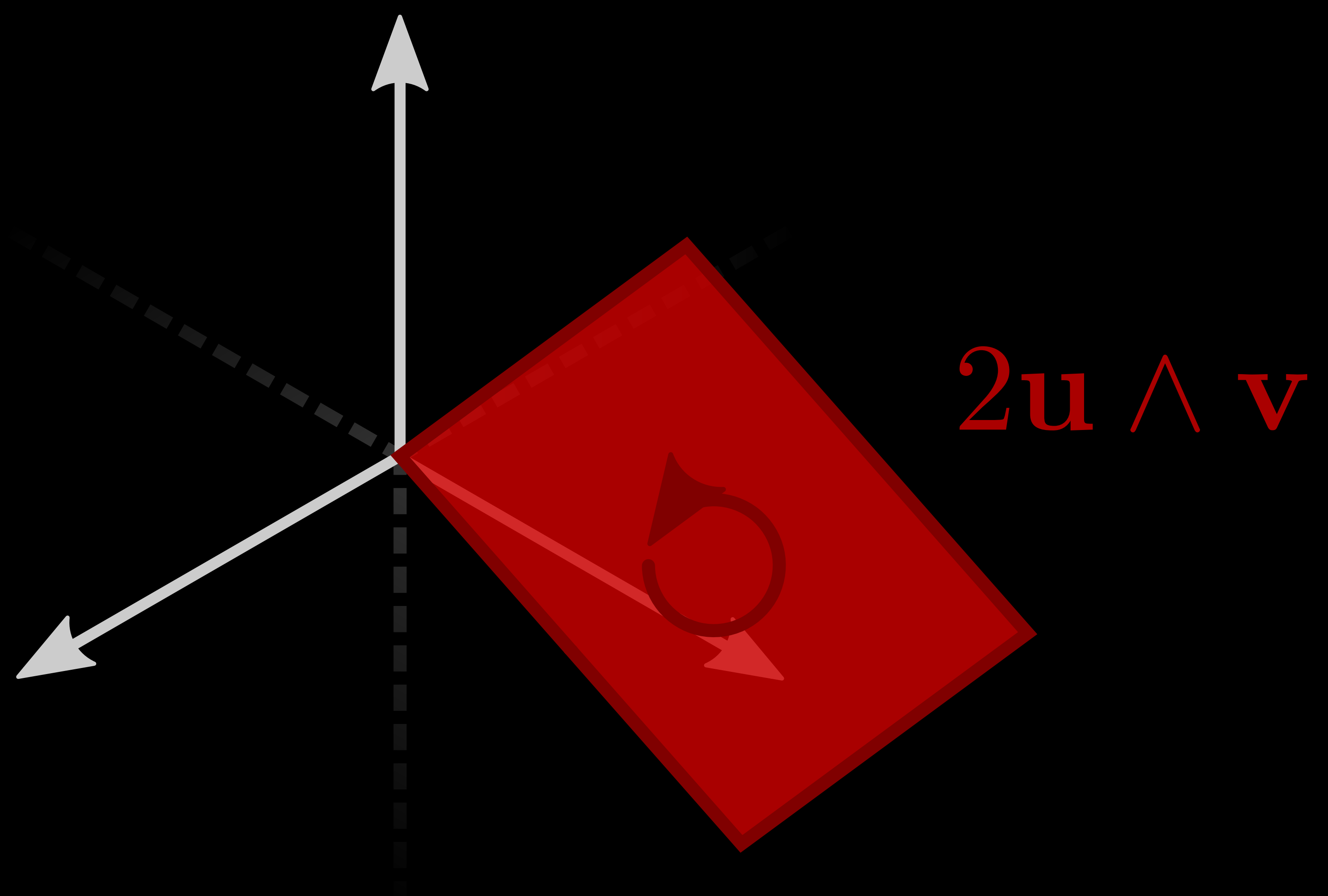


Reversing the order in outer product reverses the orientation

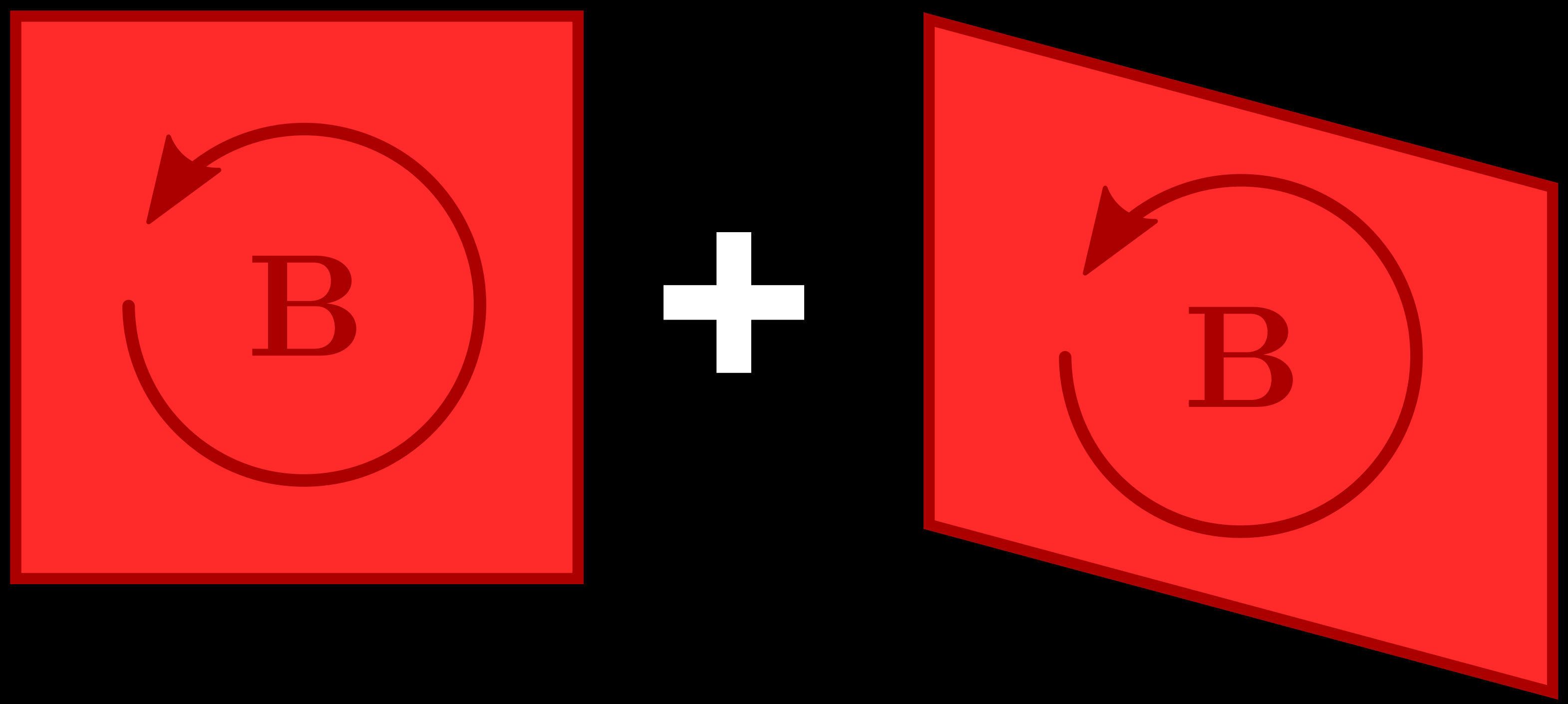




A bivector can be scaled



To add two bivectors in 2D, simply join them



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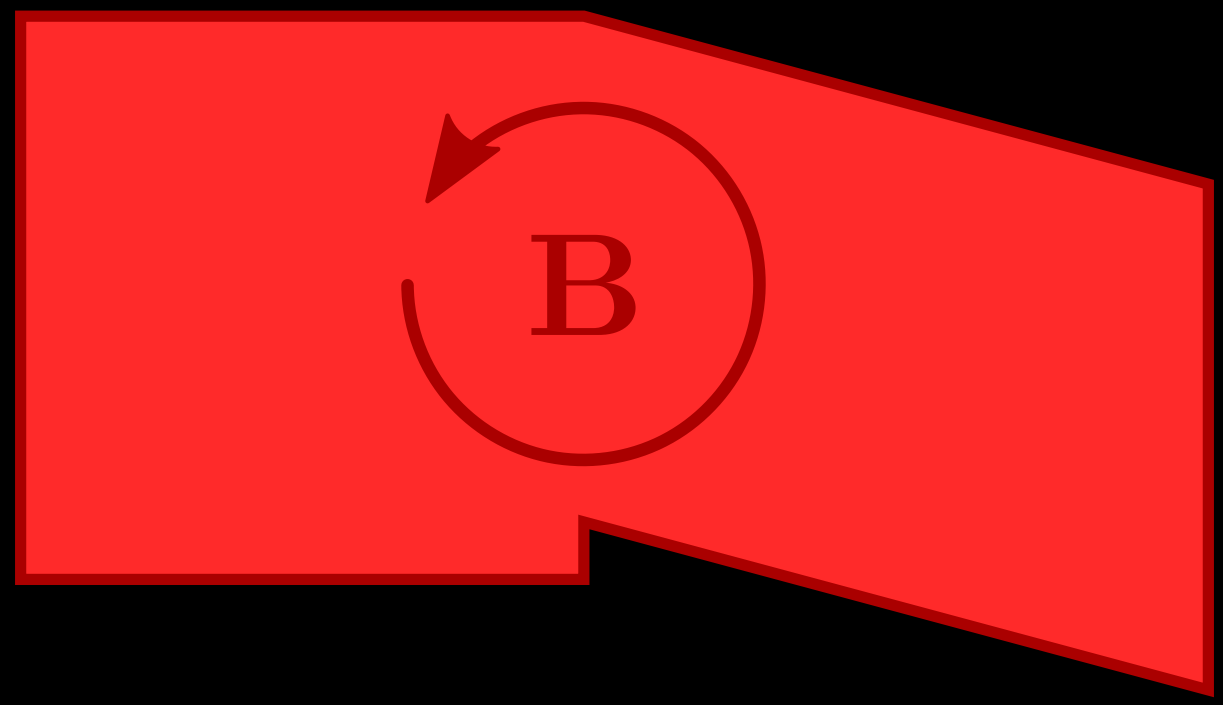
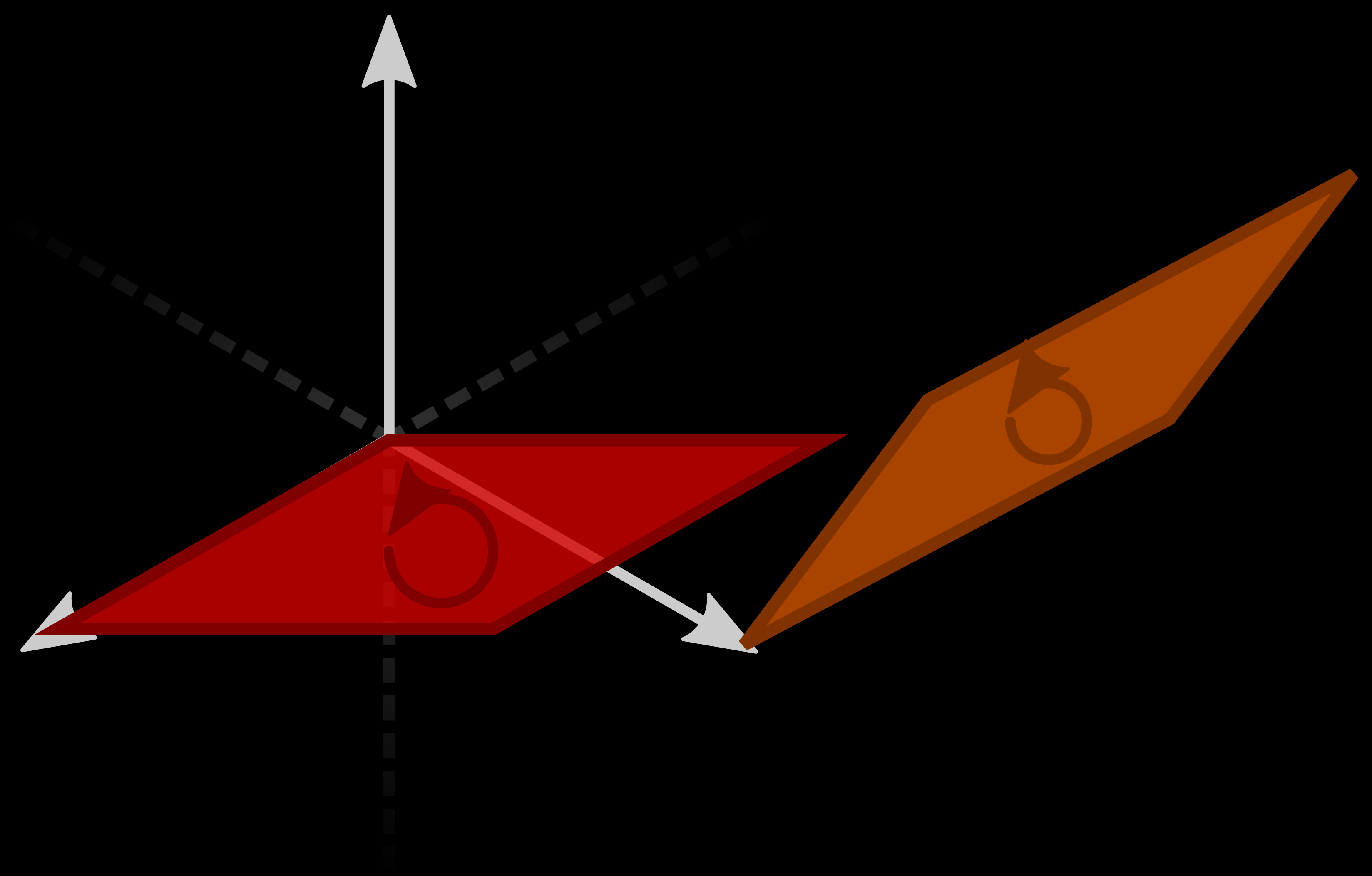


Bivector addition in 3D is more involved



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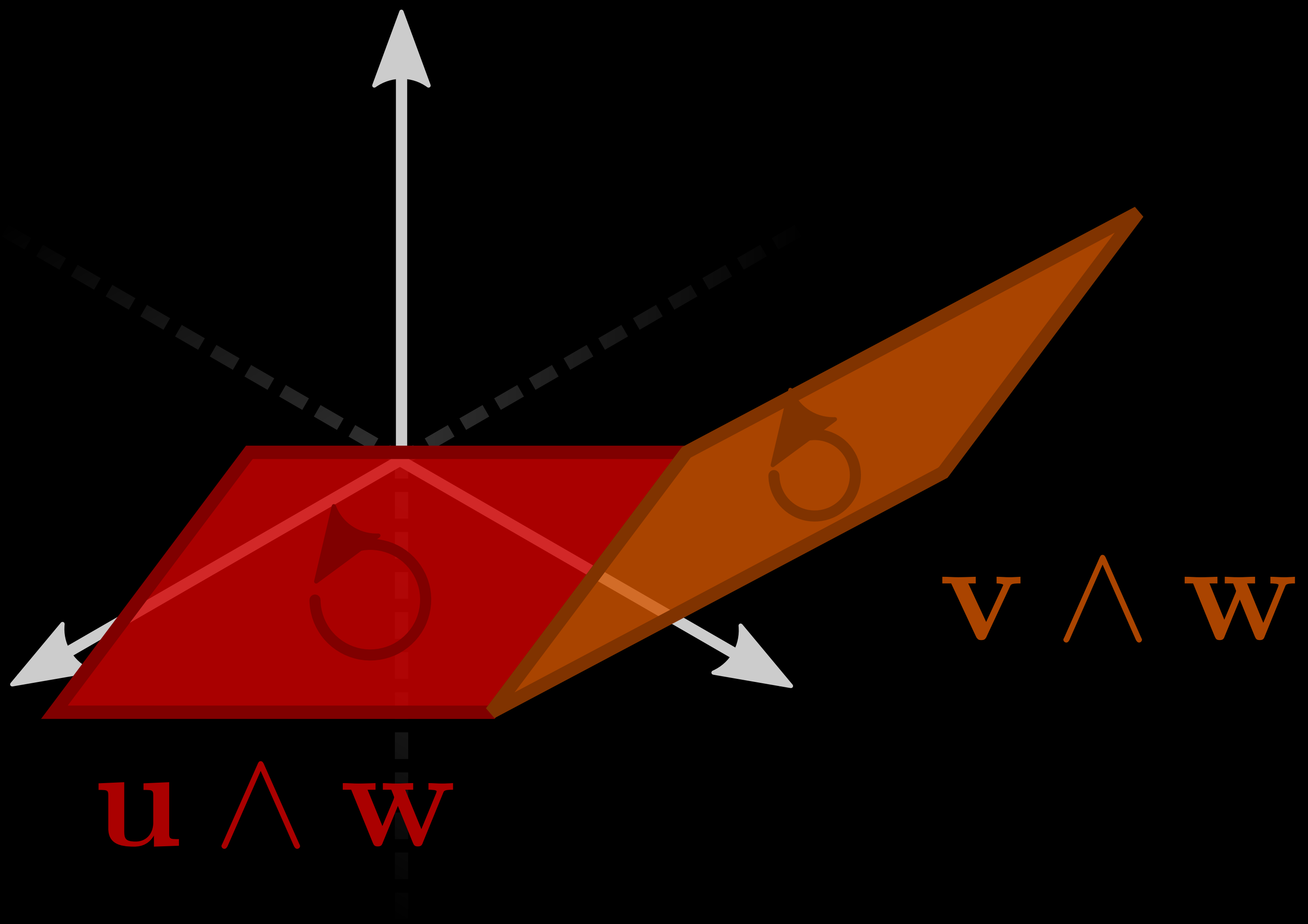

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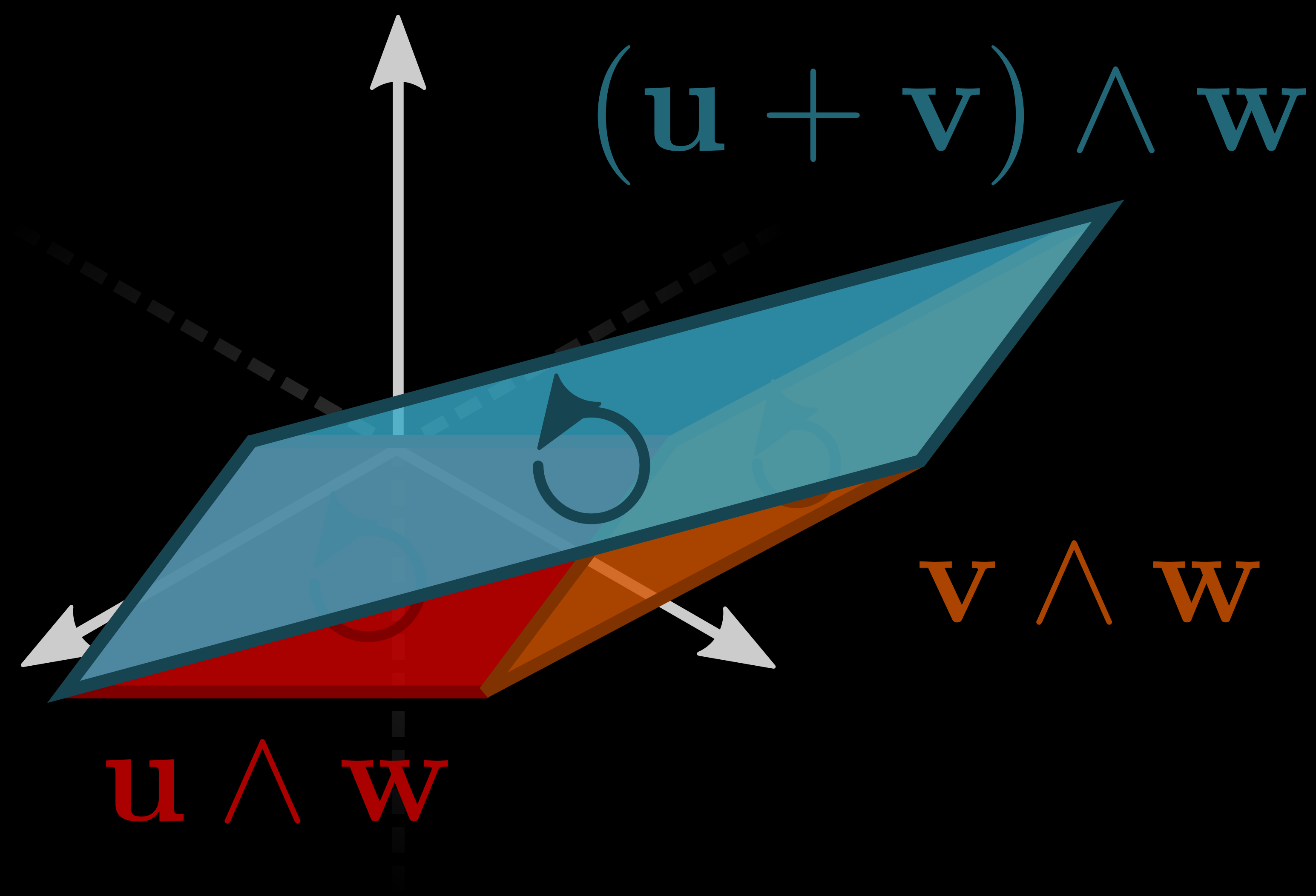


Formal definition:

The bivector space $\Lambda^2(V)$ over a vector space V is freely generated by formal products $\mathbf{u} \wedge \mathbf{v}$ for $\mathbf{u}, \mathbf{v} \in V$, such that

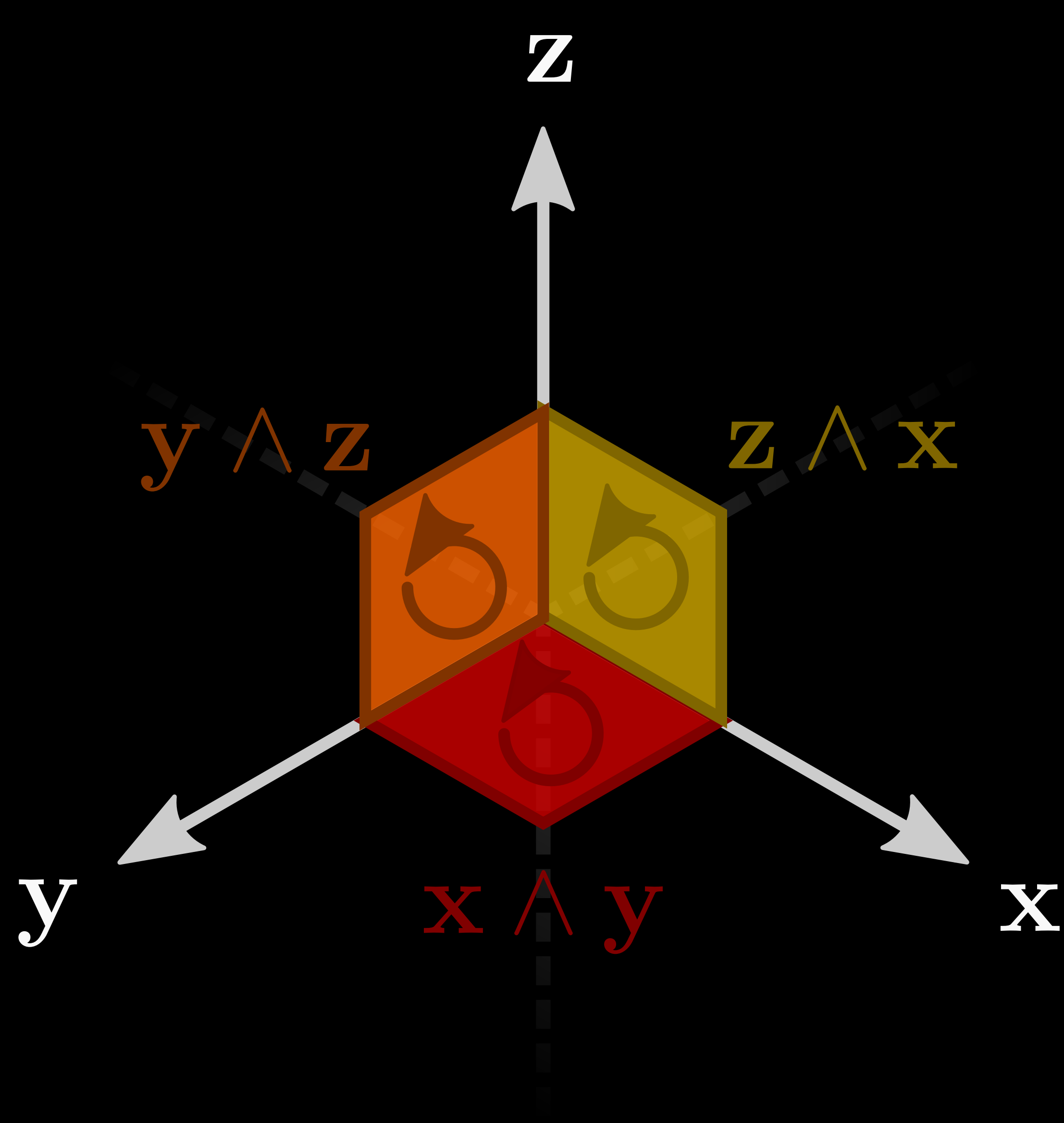
$$
\begin{aligned}
&\mathbf{u} \wedge \mathbf{u} = 0 \\
&(a\mathbf{u} + b\mathbf{v}) \wedge \mathbf{w} = a(\mathbf{u} \wedge \mathbf{w}) + b(\mathbf{v} \wedge \mathbf{w}) \\
&\mathbf{u} \wedge (a\mathbf{v} + b\mathbf{w}) = a(\mathbf{u} \wedge \mathbf{v}) + b(\mathbf{u} \wedge \mathbf{w})
\end{aligned}
$$

Similarly, one can define spaces of k-vectors for $k \leq \dim V$

WARNING

If $\dim V > 3$, not all bivectors are a result of exterior product of two vectors.
The ones that are will be called **blades**.

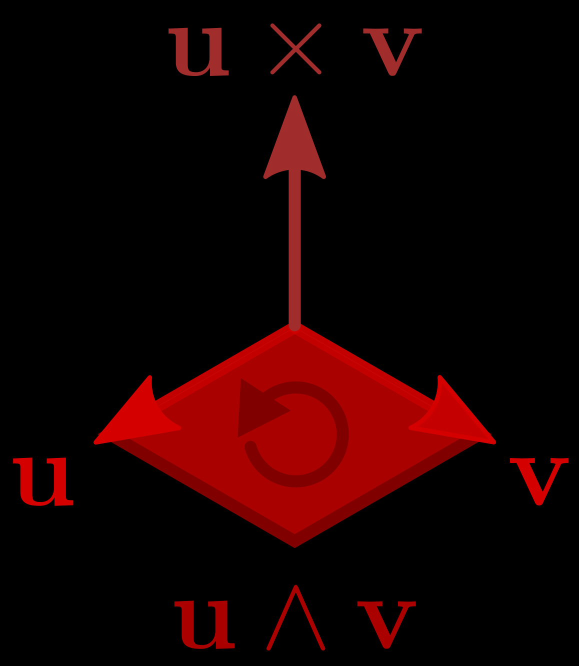
In n-dimensional space, there are $\binom{n}{2}$ basis bivectors



In $\mathbb{R}^3$, $\mathbf{u} \wedge \mathbf{v}$ is orthogonal to $\mathbf{u} \times \mathbf{v}$, with the same coordinates

$$\begin{aligned} \mathbf{u} \wedge \mathbf{v} = & \quad (u_x v_y - v_x u_y)\mathbf{x} \wedge \mathbf{y} \\ & - (u_x v_z - v_x u_z)\mathbf{z} \wedge \mathbf{x} \\ & + (u_y v_z - v_y u_z)\mathbf{y} \wedge \mathbf{z} \end{aligned}$$

$$\begin{aligned} \mathbf{u} \times \mathbf{v} = & \quad (u_x v_y - v_x u_y)\mathbf{z} \\ & - (u_x v_z - v_x u_z)\mathbf{y} \\ & + (u_y v_z - v_y u_z)\mathbf{x} \end{aligned}$$



Let V be also equipped with an inner product (so that lengths are defined).

$\mathbf{u} \cdot \mathbf{v}$	$\mathbf{u} \wedge \mathbf{v}$
inner product	exterior product

We will define the **geometric product**, so that

$$\underbrace{\mathbf{uv}}_{\text{geometric product}} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}$$

No, you can't add apples and oranges!

No, you can't add apples and oranges!

I can, in the direct sum of apple space and orange space.

The **exterior space** over V is

$$\Lambda(V) = \underset{\substack{|\\ \text{scalars}}}{\Lambda^0(V)} \oplus \underset{\substack{|\\ \text{vectors}}}{\Lambda^1(V)} \oplus \underset{\substack{|\\ \text{bivectors}}}{\Lambda^2(V)} \oplus ... \oplus \Lambda^{dim V}(V)$$

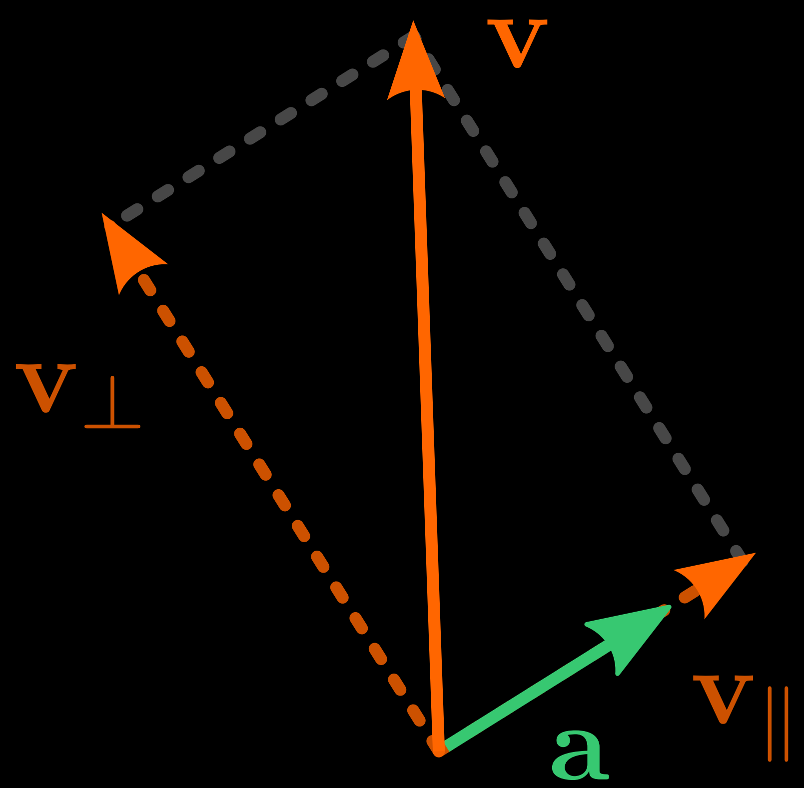
The **geometric product** is the (unique) bilinear product on $\Lambda(V)$ for which

$$\begin{aligned} (AB)C &= A(BC) \qquad && \forall A, B, C \in \Lambda(V) \\ \mathbf{v}^2 = \mathbf{v}\mathbf{v} &= ||\mathbf{v}||^2 \qquad && \forall \mathbf{v} \in V \end{aligned}$$

This structure is called geometric (or Clifford) algebra. In particular, each vector $\mathbf{v}$ has an inverse $\mathbf{v}^{-1} = \frac{\mathbf{v}}{||\mathbf{v}||^2}$ and $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u})$

Reflection by a unit vector $\mathbf{a}$

(only the part perpendicular to $\mathbf{a}$ is reflected)



$$\mathbf{v} \quad = \quad \mathbf{v}_\parallel - \mathbf{v}_\perp$$

Reflection by a unit vector $\mathbf{a}$

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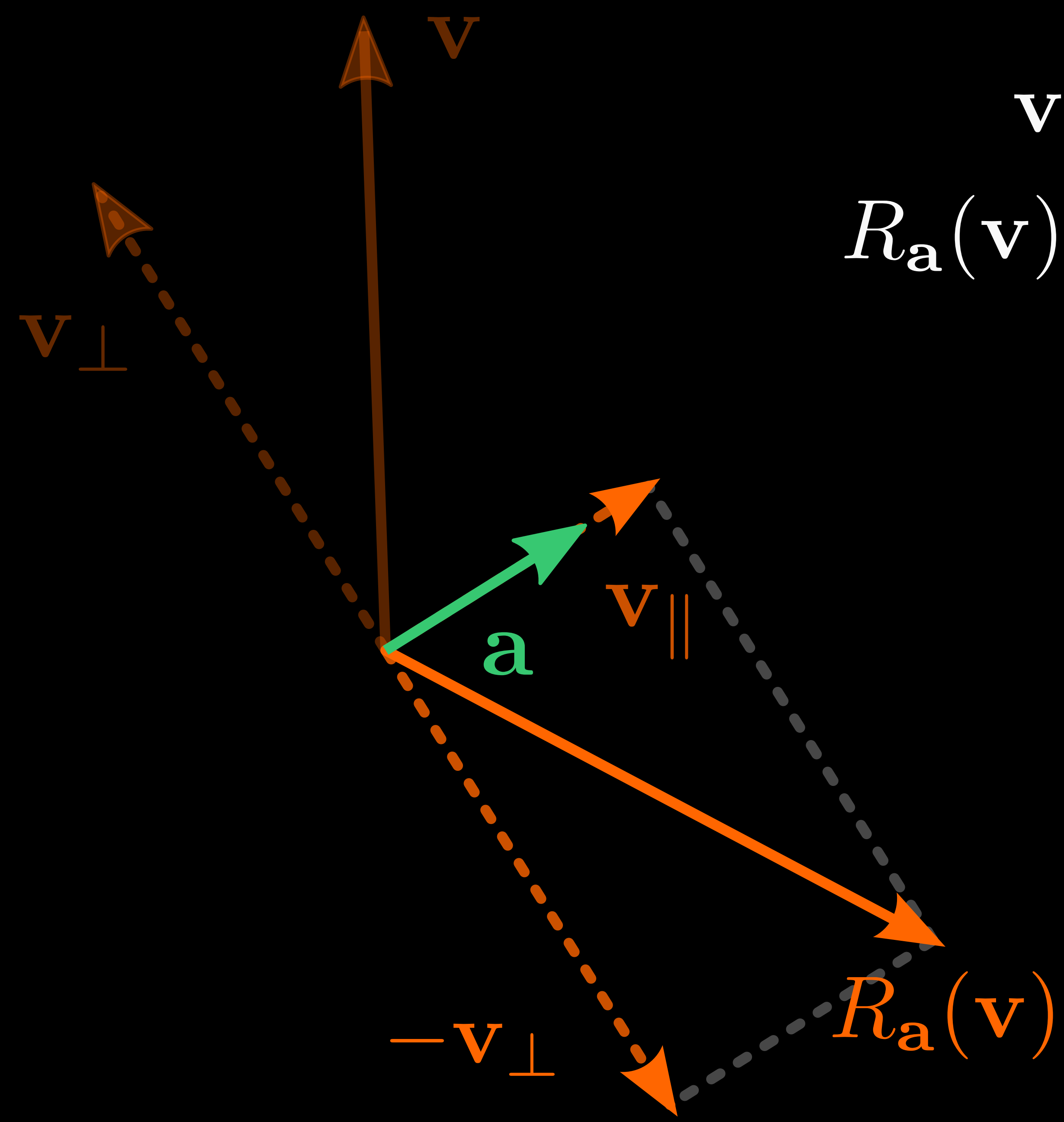
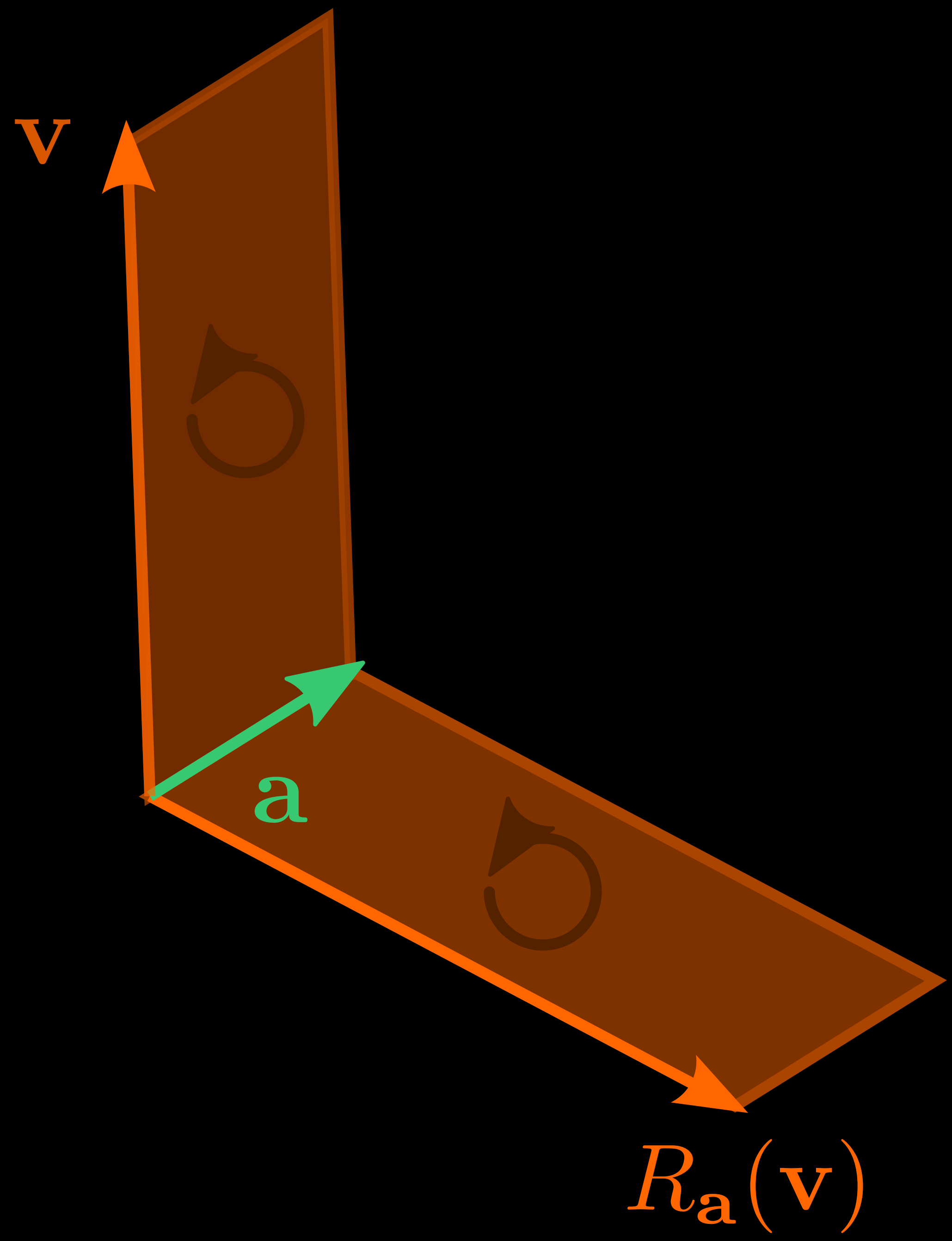


$$
\begin{aligned}
\mathbf{v} &= \mathbf{v}_\parallel - \mathbf{v}_\perp \\
R_\mathbf{a}(\mathbf{v}) &= \mathbf{v}_\parallel - \mathbf{v}_\perp \\
&= (\mathbf{v}\cdot\mathbf{a})\mathbf{a} - (\mathbf{v} - (\mathbf{v}\cdot\mathbf{a})\mathbf{a}) \\
&= 2(\mathbf{v}\cdot\mathbf{a})\mathbf{a} - \mathbf{v} \\
&= 2(\frac{1}{2}(\mathbf{va}+\mathbf{av}))\mathbf{a} - \mathbf{v} \\
&= \mathbf{va}^2 + \mathbf{ava} - \mathbf{v} \\
&= \mathbf{ava}
\end{aligned}
$$

Reflection by a unit vector $\mathbf{a}$ - another look

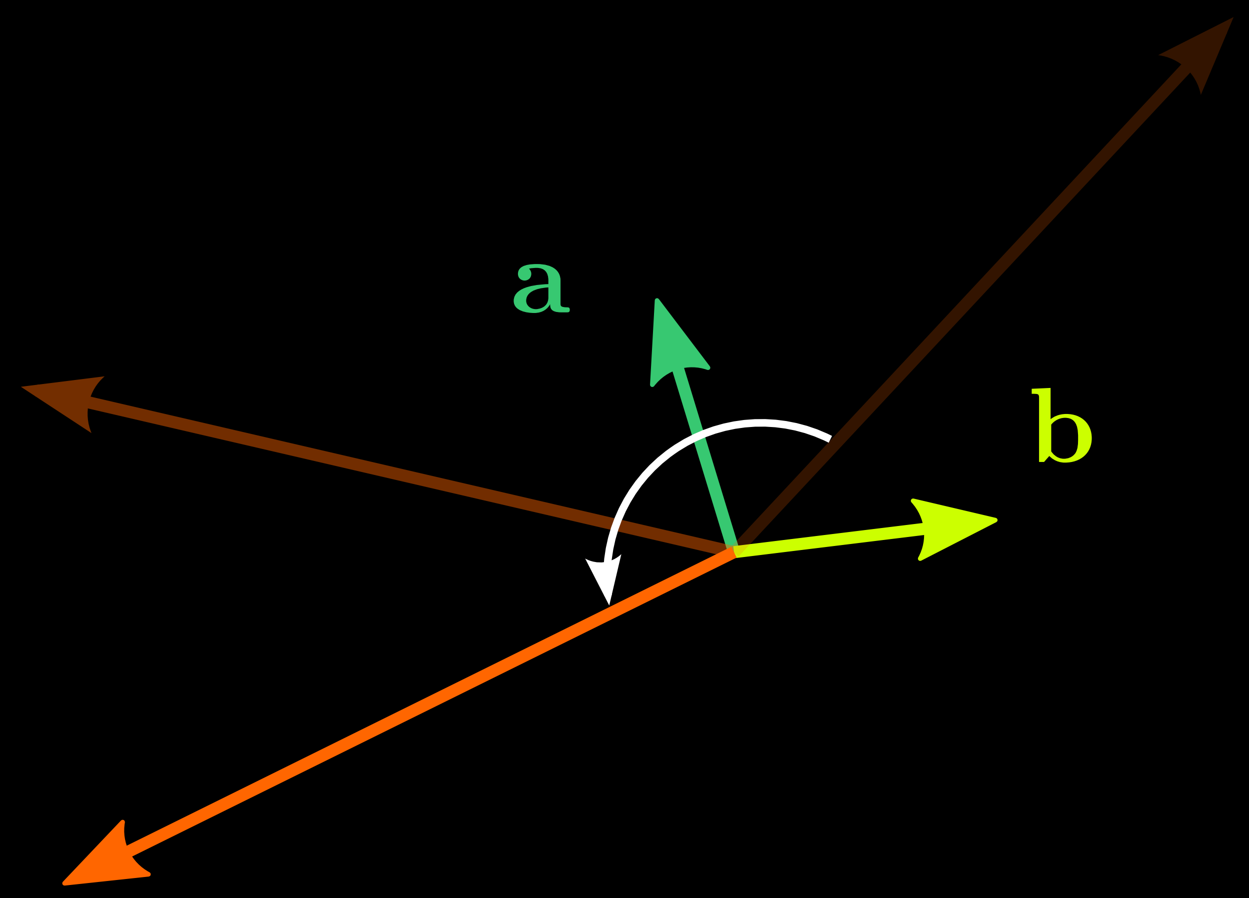
$R_{\mathbf{a}}(\mathbf{v}) \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{v}$ and $R_{\mathbf{a}}(\mathbf{v}) \wedge \mathbf{a} = \mathbf{a} \wedge \mathbf{v}$



$$\begin{aligned} R_{\mathbf{a}}(\mathbf{v})\mathbf{a} &= \mathbf{av} \\ R_{\mathbf{a}}(\mathbf{v}) &= \mathbf{ava} \end{aligned}$$

Two reflections make a rotation. For $\mathbf{a}, \mathbf{b}$ unit vectors, $\mathbf{v} \mapsto \mathbf{bavab}$ rotates in their plane by twice their angle.

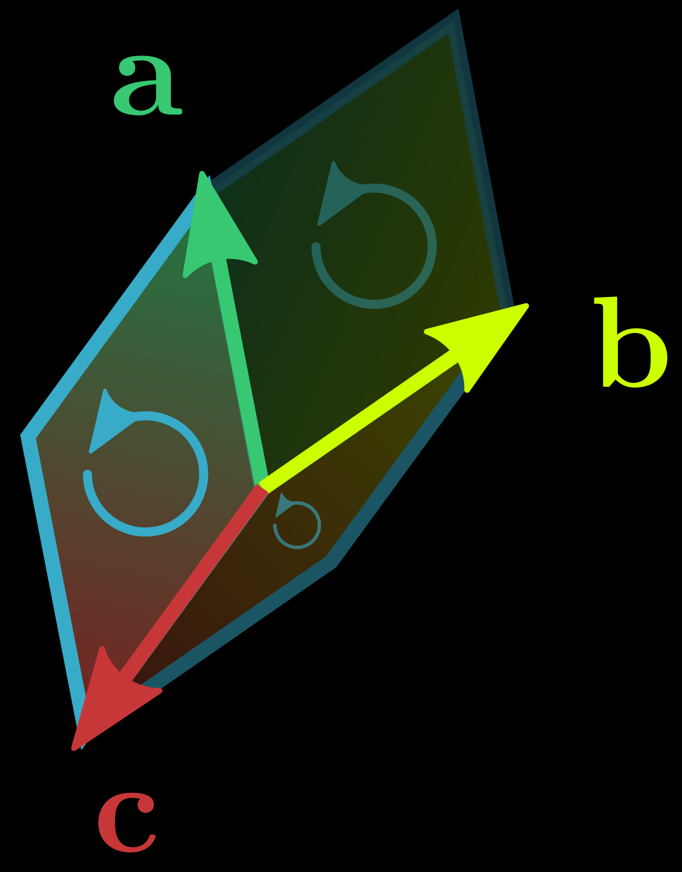
Geometric products of unit vectors are called **versors** or **pinors**. Geometric products of even number of unit vectors are called **rotors** or **spinors**.



When two rotation planes intersect, composition of rotations in them is another rotation

If the $\mathbf{b}$ is the intersection unit vector, we can find vectors $\mathbf{a}$, $\mathbf{c}$, such that the rotors are $\mathbf{ab}$, $\mathbf{bc}$

Their composition is then $\mathbf{abbc} = \mathbf{ac}$

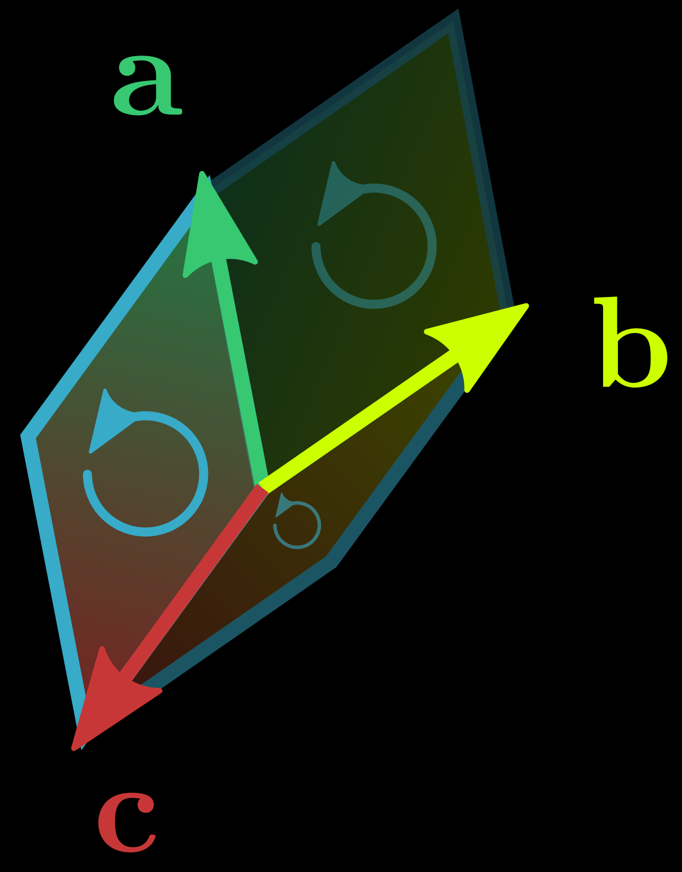


Rotations don't always commute

The composition of $\mathbf{ab}$, $\mathbf{bc}$ is $\mathbf{abbc} = \mathbf{ac}$

When done in opposite order, the composition is $\mathbf{bcab}$

The resulting rotation plane is reflected by $\mathbf{b}$ and the sense is reversed



If $\mathbf{u}, \mathbf{v}$ are orthonormal, $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = \mathbf{u} \wedge \mathbf{v}$.
So $\mathbf{vu} = \mathbf{v} \wedge \mathbf{u} = -\mathbf{u} \wedge \mathbf{v} = -\mathbf{uv}$.
Thus $(\mathbf{uv})(\mathbf{uv}) = \mathbf{u}(\mathbf{vu})\mathbf{v} = -\mathbf{uuvv} = -1$.
That means for every unit blade $\mathbf{I}$, $\mathbf{I}^2 = -1$.

In particular, since there is only one bivector in the plane, $\Lambda^0(\mathbb{R}^2) \oplus \Lambda^2(\mathbb{R}^2)$ is isomorphic to $\mathbb{C}$ as a real algebra

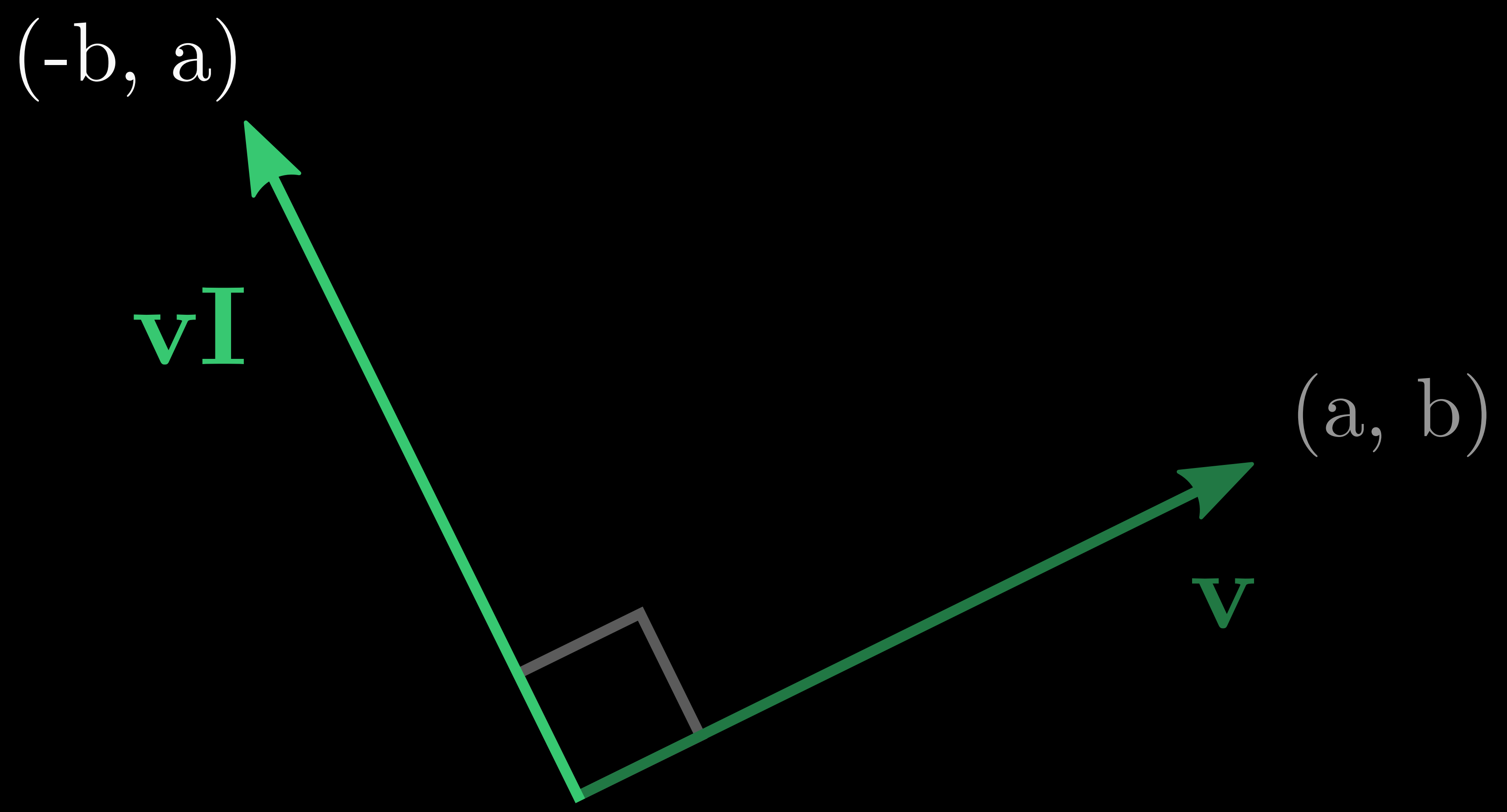
For $S \in \Lambda(V)$, we define $e^S = \sum_{n=0}^{\infty} \frac{S^n}{n!}$

If $\mathbf{I}$ is a unit blade, then $e^{\phi \mathbf{I}} = \cos\phi + \mathbf{I}\sin\phi$

Rotation in the plane

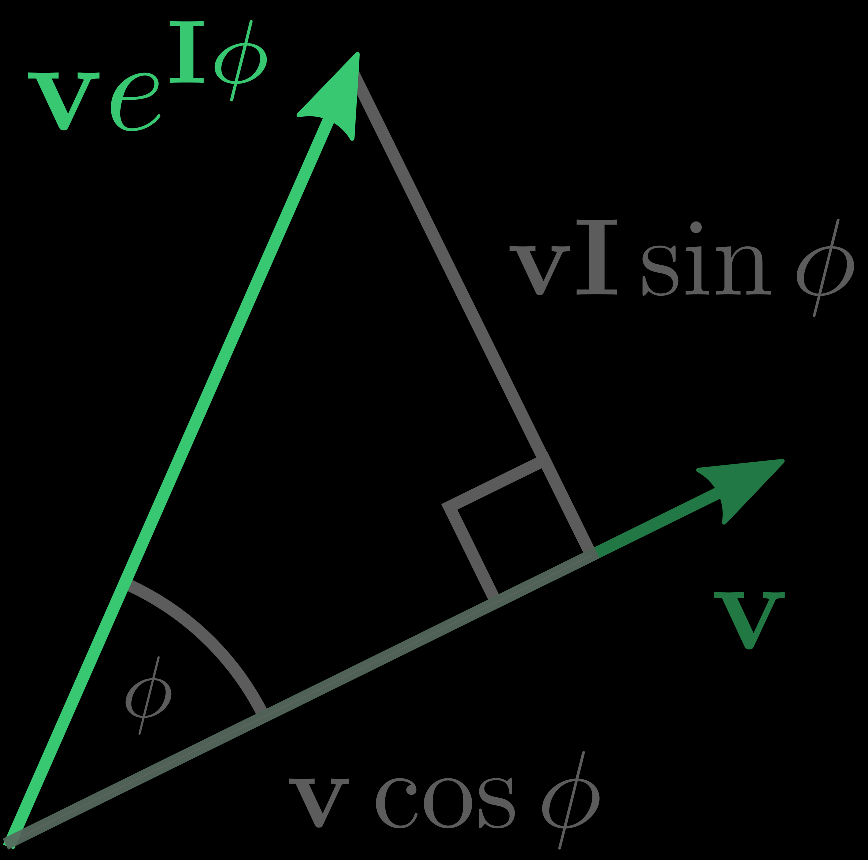
Right multiplication by $\mathbf{I}$ rotates by $\frac{\pi}{2}$ (90°)

Given an orthonormal basis $\mathbf{x}, \mathbf{y}$, so that $\mathbf{I} = \mathbf{xy}$, $(a\mathbf{x} + b\mathbf{y})\mathbf{I} = (-b\mathbf{x} + a\mathbf{y})$



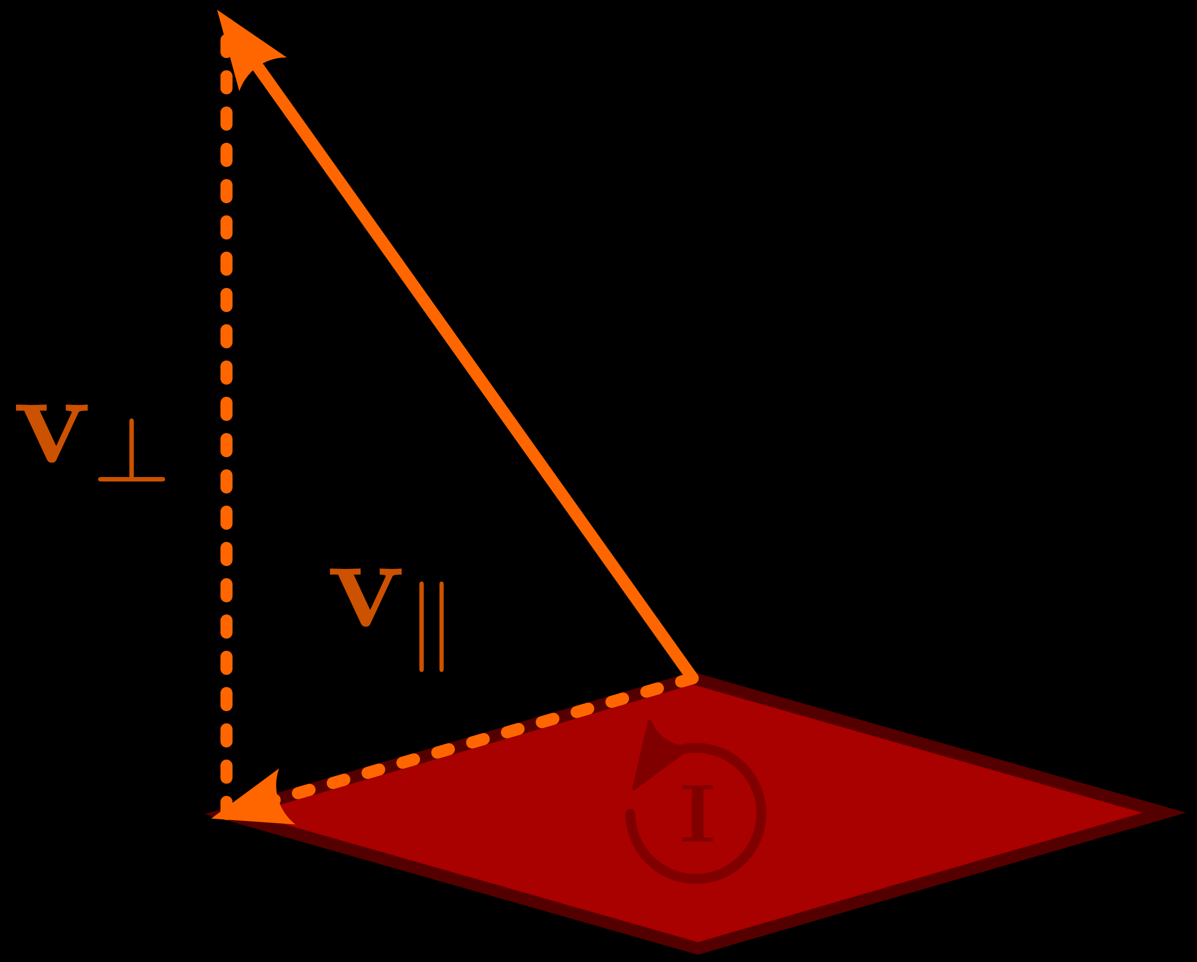
Rotation in the plane

Then, right multiplication by $e^{\mathbf{I}\phi}$ rotates by ϕ



Rotation in the general space

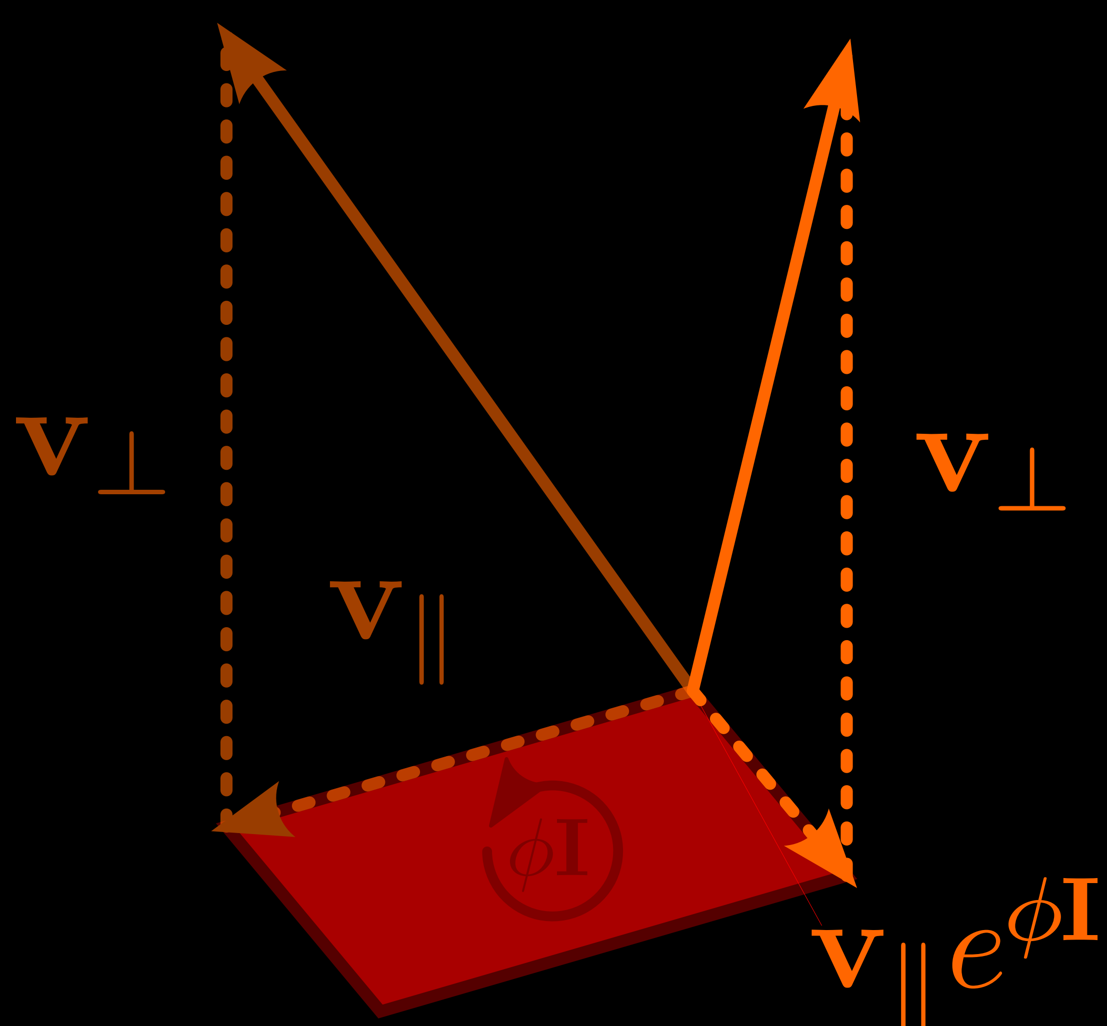
(only the part parallel to the rotation plane **I** gets rotated)



$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$$

Rotation in the general space

(only the part parallel to the rotation plane $\mathbf{I}$ gets rotated)



$$
\begin{aligned}
\mathbf{v} &= \mathbf{v}_\parallel + \mathbf{v}_\perp \\
\rho_{\phi\mathbf{I}}(\mathbf{v}) &= \mathbf{v}_\parallel e^{\phi\mathbf{I}} + \mathbf{v}_\perp \\
&= \mathbf{v}_\parallel e^{\frac{\phi}{2}\mathbf{I}} e^{\frac{\phi}{2}\mathbf{I}} + \mathbf{v}_\perp e^{-\frac{\phi}{2}\mathbf{I}} e^{\frac{\phi}{2}\mathbf{I}} \\
&= e^{-\frac{\phi}{2}\mathbf{I}} \mathbf{v}_\parallel e^{\frac{\phi}{2}\mathbf{I}} + e^{-\frac{\phi}{2}\mathbf{I}} \mathbf{v}_\perp e^{\frac{\phi}{2}\mathbf{I}} \\
&= e^{-\frac{\phi}{2}\mathbf{I}} \mathbf{v} e^{\frac{\phi}{2}\mathbf{I}}
\end{aligned}
$$

That's exactly the formula for quaternion rotation! In fact, let $\mathbf{I} = \mathbf{xy}, \mathbf{J} = \mathbf{yz}, \mathbf{K} = \mathbf{x}$. Then

$$\begin{aligned} \mathbf{I}^2 &= -1 & \mathbf{IJ} &= \mathbf{K} \\ \mathbf{J}^2 &= -1 & \mathbf{JK} &= \mathbf{I} \\ \mathbf{K}^2 &= -1 & \mathbf{KI} &= \mathbf{J} \end{aligned}$$

$\Lambda^0(\mathbb{R}^3) \oplus \Lambda^2(\mathbb{R}^3)$ is isomorphic to quaternions as an algebra

Thank you