Fundamental group

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Motivation

Brouwer's fixed point theorem: informal

We cannot ever perfectly mix coffee.

Brouwer's fixed point theorem: rigorous

We cannot ever perfectly mix coffee.

There is no continuous map $\mathbb{D}^2 \to \mathbb{D}^2$ without fixed point.

Here $\mathbb{D}^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$

Brouwer's fixed point theorem: formal

We cannot ever perfectly mix coffee.

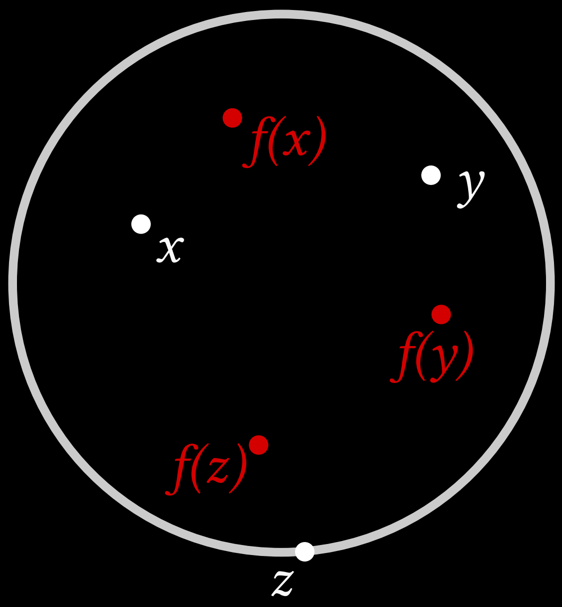
There is no continuous map $\mathbb{D}^2 \to \mathbb{D}^2$ without fixed point.

$\forall f : \mathbb{D}^2 \to \mathbb{D}^2 \exists x \in \mathbb{D}^2 f(x) = x$

Brouwer's fixed point theorem: start of a proof

Supposе there is a continuous map $\mathbb{D}^2 \to \mathbb{D}^2$ without fixed point.

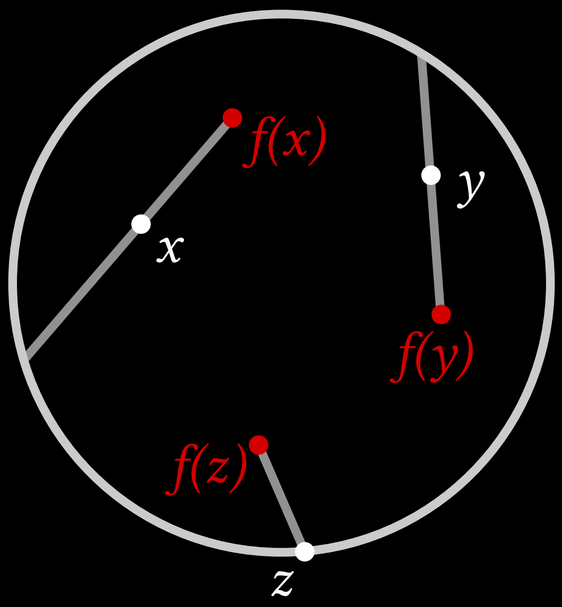
$\forall x \in \mathbb{D}^2 \, f(x) \neq x$



Brouwer's fixed point theorem: start of a proof

Suppose there is a continuous map $\mathbb{D}^2 \to \mathbb{D}^2$ without fixed point.

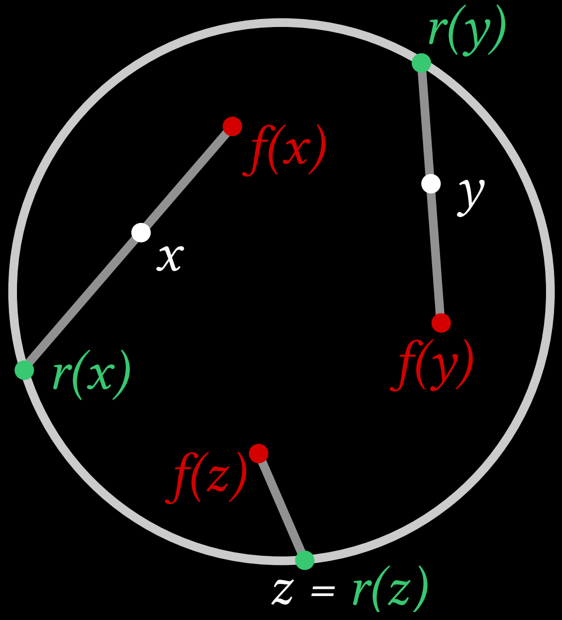
Follow rays from $f(x)$ to x until they intersect the boundary circle $\mathbb{S}^1$.



Brouwer's fixed point theorem: start of a proof

Follow rays from $f(x)$ to x until they intersect the boundary circle $\mathbb{S}^1$.

This gives a continuous map $r : \mathbb{D}^2 \to \mathbb{S}^1$ which is an identity on $\mathbb{S}^1$.



Topology

A general study of continuous maps

What is a *space*?

In this lecture: a subset of $\mathbb{R}^n$

In the wild nature: *topological space* (a very abstract and general definition)

From now on, *map* will mean **continuous** function.

What is a topological equivalence?

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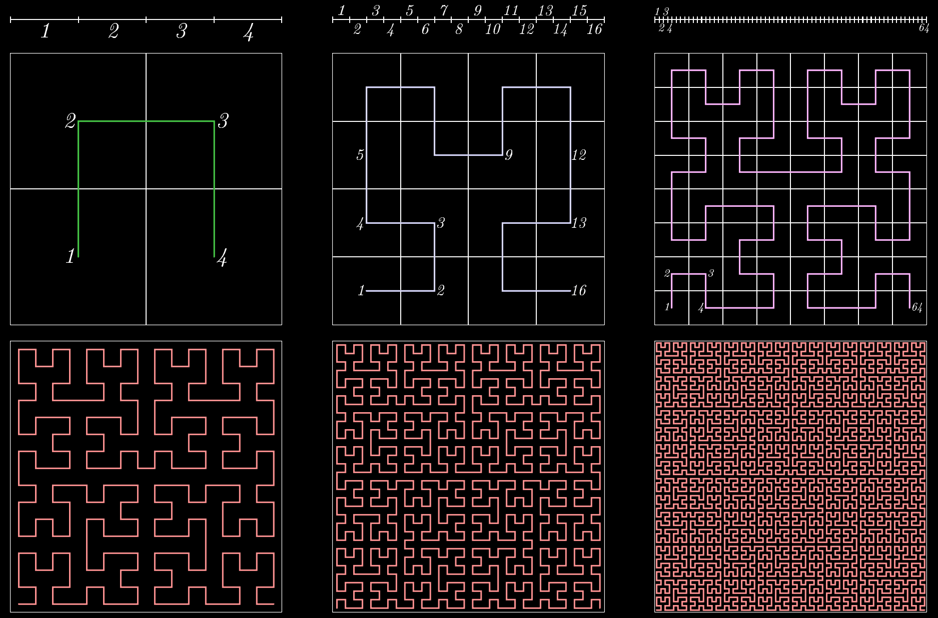
Injective and surjective continuous map?

What is a topological equivalence?

Injective and surjective continuous map? Not good.

Hilbert curve: a curve that fills the whole square



What is a topological equivalence?

A map $f : S \to T$ is called a *homeomorphism* if it has a two-sided continuous inverse, i.e. there is a map $g : T \to S$ such that both composites fg and gf are identity.
Then S and T are called homeomorphic (just a fancy word for topologically equivalent).

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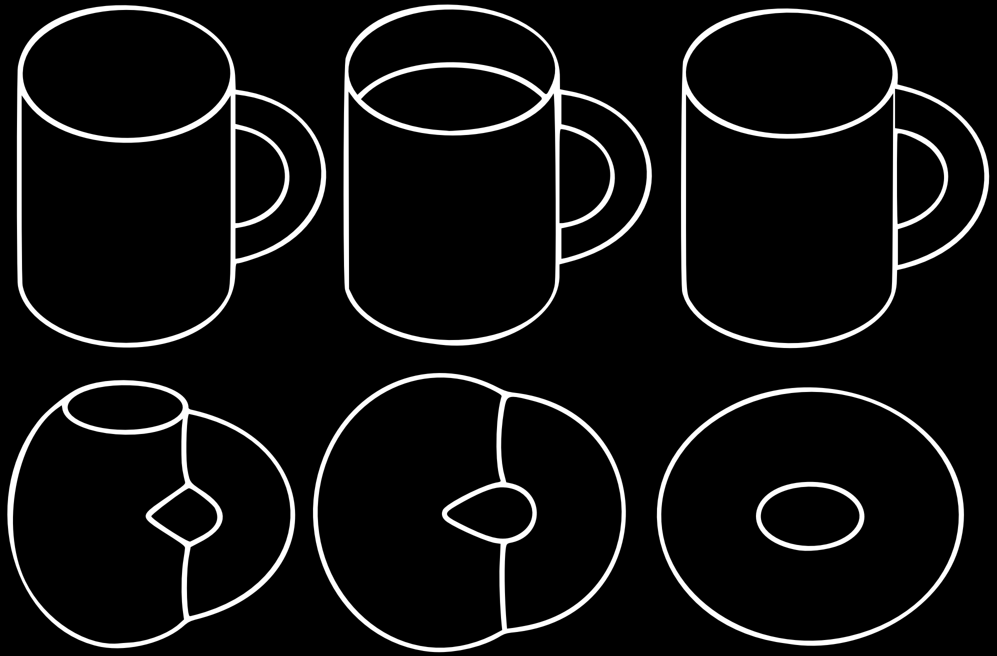
Example: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$ nor to $\mathbb{S}^1$.

Example: a topologist doesn't know a difference between a cup and a donut



Is the open unit disk homeomorphic to $\mathbb{R}^2$?

$$\mathbb{D}_o^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$$

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Yes, the homeomorphism is given in polar coordinates by
$(r, \varphi) \mapsto (\tan(\frac{\pi}{2} r), \varphi)$

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Yes, the homeomorphism is given in polar coordinates by $(r, \varphi) \mapsto (\tan(\frac{\pi}{2} r), \varphi)$

Analogous result holds for $\mathbb{R}^n$.

Is $\mathbb{S}^n$ homeomorphic to $\mathbb{R}^n$?

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No; not completely trivial to prove.

Is $\mathbb{S}^n$ with one point removed homeomorphic to $\mathbb{R}^n$?

e. g. $\mathbb{S}^n \setminus \{(0,0,1)\}$

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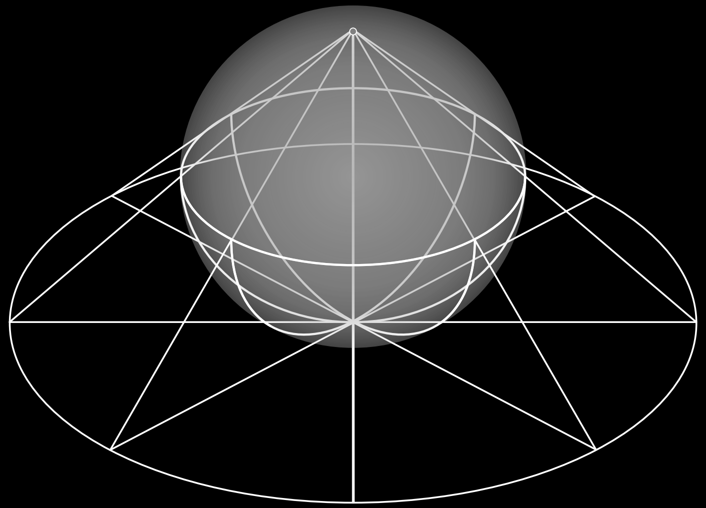
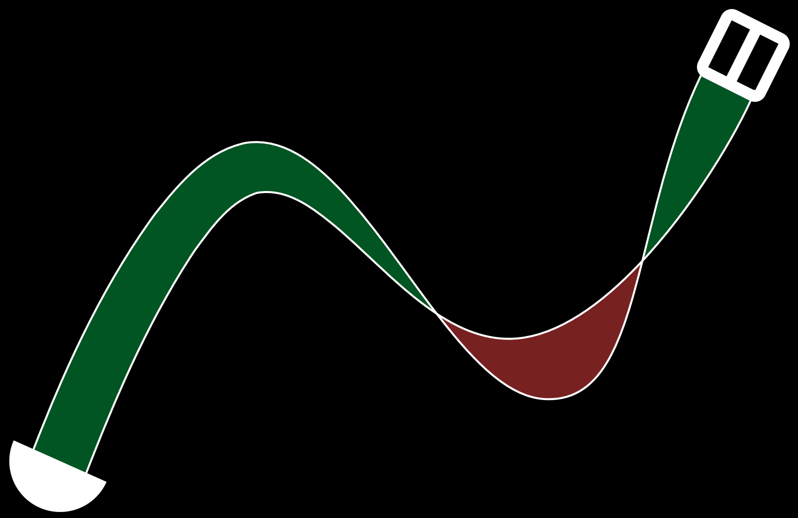
Yes, the homeomorphism is given by the stereographic projection.



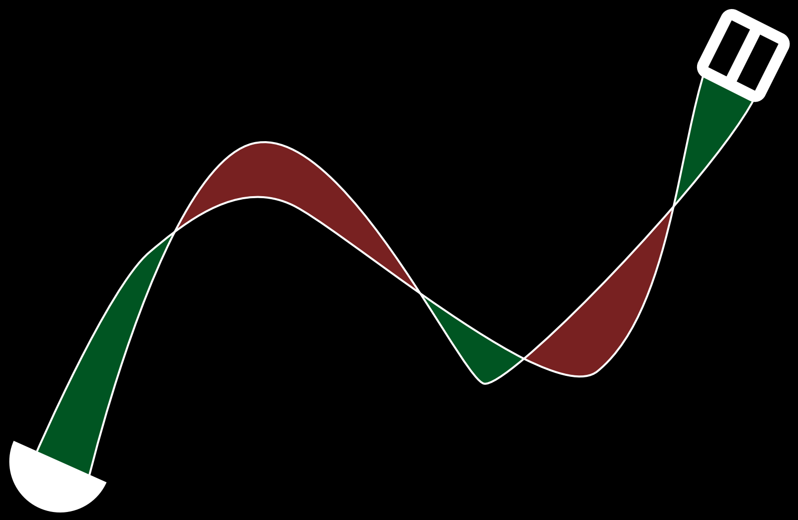
Dirac belt trick

Have a belt with one end fixed and one twist.
It **cannot** be straightened without rotating the buckle.



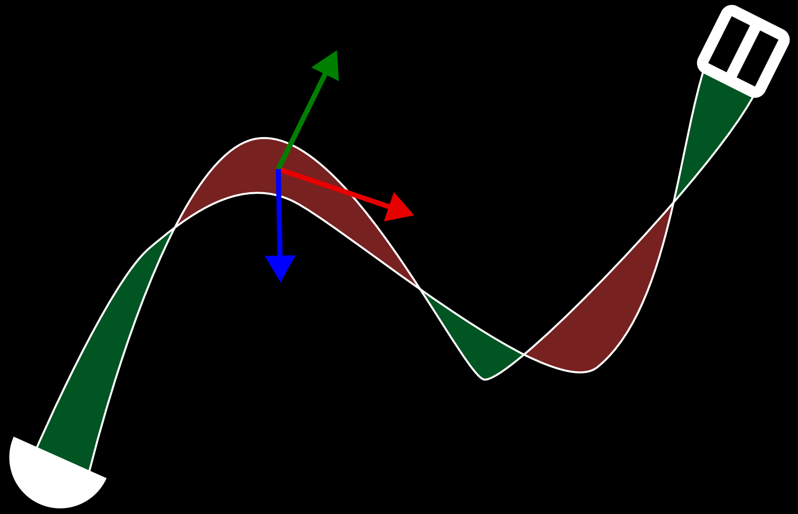
Dirac belt trick

Have a belt with one end fixed and **two** twists.
It **can** be straightened without rotating the buckle!



Dirac belt trick: what's going on?

As we traverse the belt, track the unit vectors:
(in the belt direction; to the side; perpendicular to the belt)

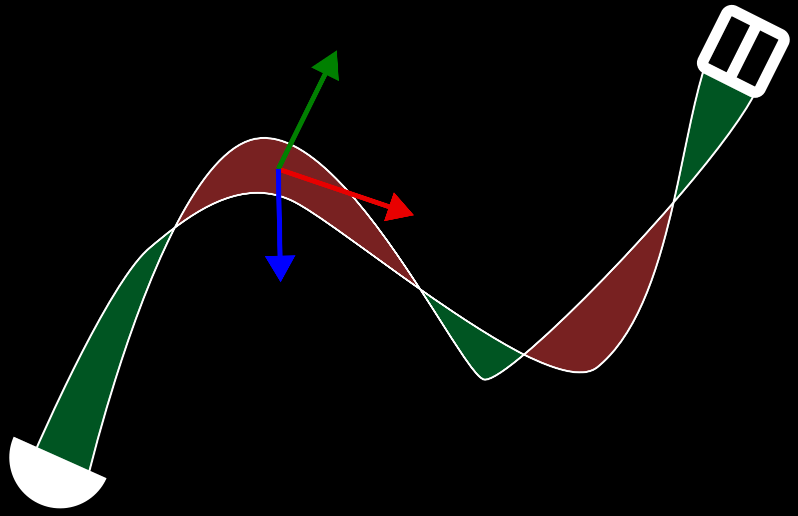


Dirac belt trick: what's going on?

As we traverse the belt, track the unit vectors:

(in the belt direction; to the side; perpendicular to the belt)

We get a path in the space of orthogonal vector triples (subset of $\mathbb{R}^{3\times 3}$)



Space of orthonormal positively oriented triples in $\mathbb{R}^3$

Elements may be written as matrices.

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$$

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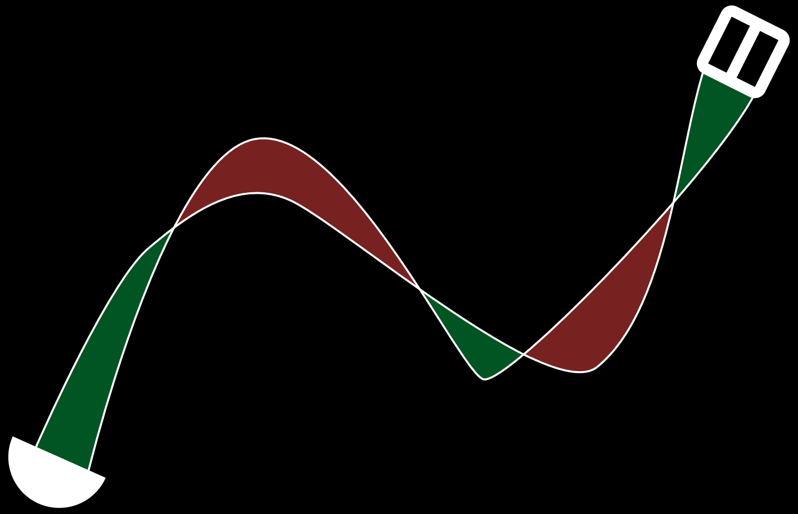
Space of orthonormal positively oriented triples in $\mathbb{R}^3$

Elements may be written as matrices. These are the rotation matrices. This space of rotations is called $SO(3)$ (*special orthogonal* group).

$$\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}$$

The belt corresponds to a path in $SO(3)$

Path: a continuous function $[0,1] \to SO(3)$



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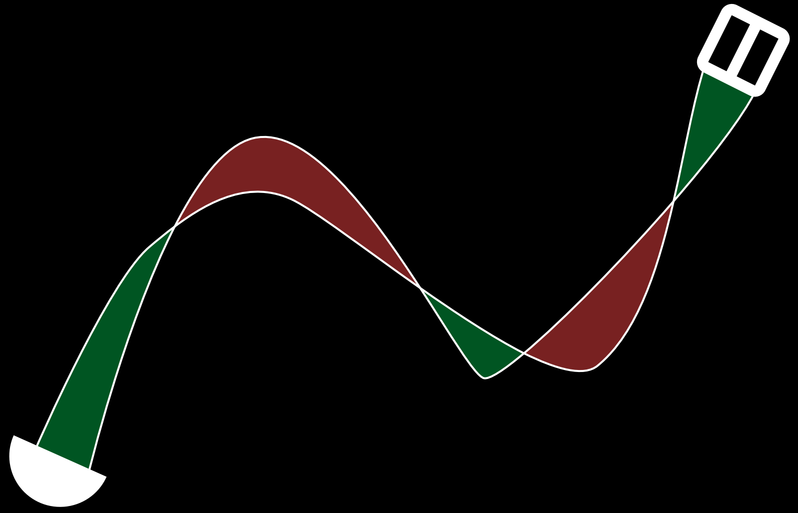

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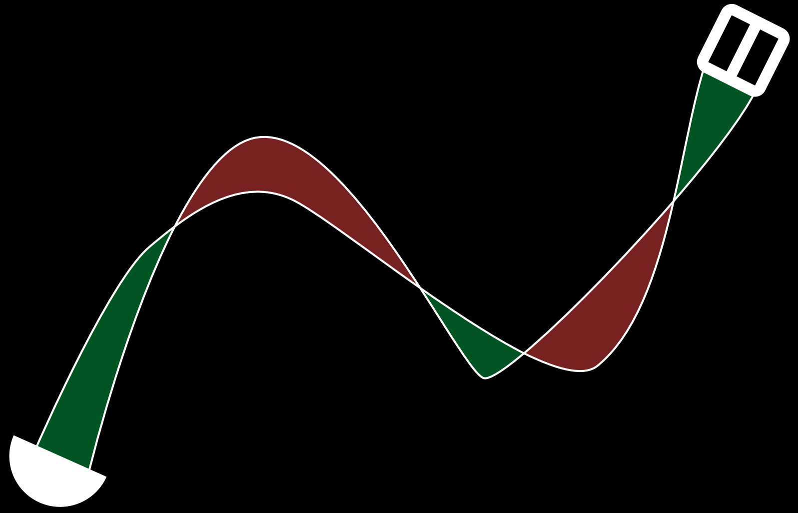
Fixing the ends of belt amounts to fixing the endpoints of the path.

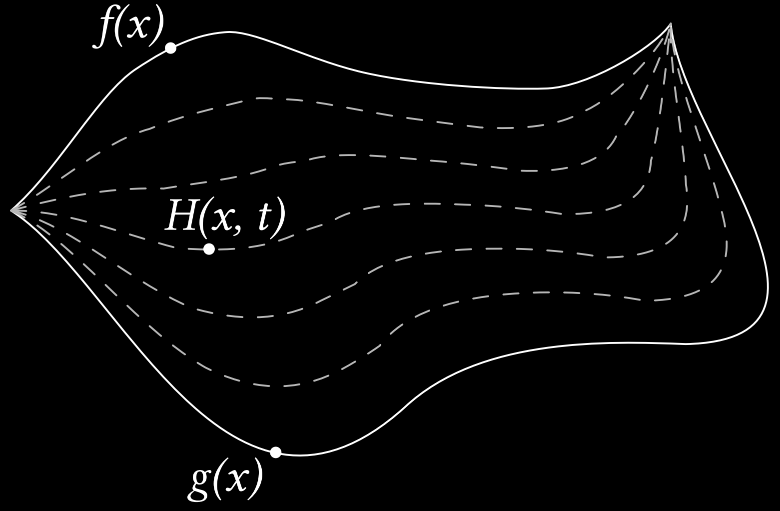
Moving the belt amounts to deforming the path.



Homotopies: deformations formally

Let $f, g : S \to T$ be maps between spaces. They are called *homotopic* if there is a map $H : S \times [0, 1] \to T$ such that for all $x \in S$:

- $H(x, 0) = f(x)$
- $H(x, 1) = g(x)$



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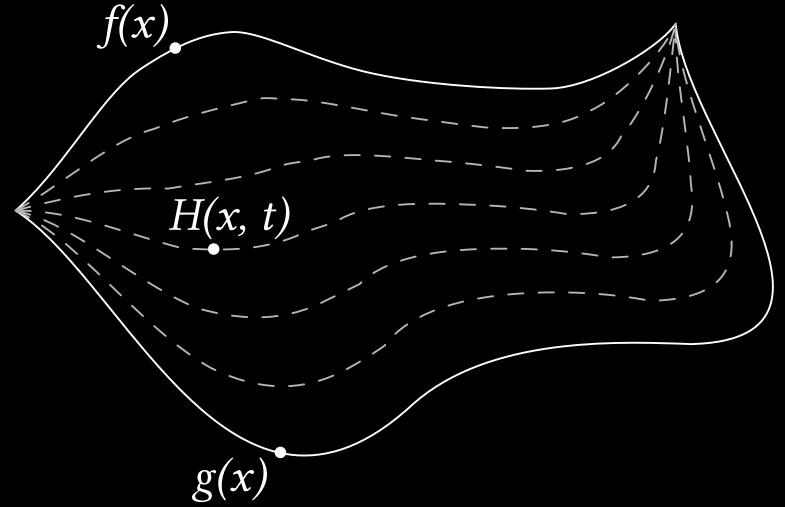
H is called *homotopy*.



Homotopy: an example

Identity on $\mathbb{R}^n$ is homotopic to a constant map to origin via the map

$$H(x, t) = tx$$

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Example: $\mathbb{R}^n$ is homotopy equivalent to a point. Such a space will be called *contractible*.

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Example: $\mathbb{R}^n$ is homotopy equivalent to a point. Such a space will be called *contractible.*

Example: Circle is homotopy equivalent to annulus.

Loop space

Let $* \in S$. Define $\Omega(S, *) = \{\rho : [0,1] \to S \mid \rho(0) = \rho(1) = *\}$.

This is a set of all loops in S beginning and ending in $*$.

Product of loops

$$\Omega(S, *) = \{\rho : [0, 1] \to S \mid \rho(0) = \rho(1) = *\}$$

For $\rho, \tau \in \Omega(S, *)$, define their product $\rho\tau : [0, 1] \to S$ by:

- $\rho\tau(t) = \rho(2t)$ for $t \in \left[0, \frac{1}{2}\right]$
- $\rho\tau(t) = \tau\left(2\left(t - \frac{1}{2}\right)\right)$ for $t \in \left[\frac{1}{2}, 1\right]$

We just go around the first loop and then around the second one.

The group axioms in $\Omega(S, *)$ hold only up to homotopy (constant at the point $*$)

- $(\rho\sigma)\tau \sim \rho(\sigma\tau)$
- $e\rho \sim \rho \sim \rho e$
- $\rho\rho^{-1} \sim e \sim \rho^{-1}\rho$

where

$e(t) = *$

$\rho^{-1}(t) = \rho(1 - t)$

Solution: factor $\Omega(S,*)$ by homotopies constant at $*$.

$$\pi_1(S,*) = \Omega(S,*)/\sim$$

It is called the *fundamental group* of S at $*$.

Aside: what if we remembered all the homotopies?

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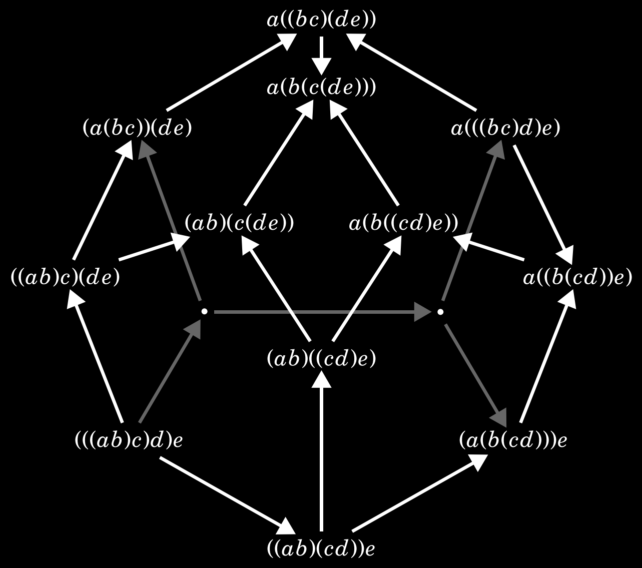
There are shapes called associahedra tracking the higher homotopies.
The corresponding algebraic object is called A_∞ algebra.



$\pi_1(S, *)$ for S contractible and any $* \in S$ is the trivial group 1.

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E.g. $\pi_1(\mathbb{R}^n, 0) = 1$.

A space S is called *path connected* if there is a path between any two of its points.

For all $a, b \in S$, there is $\varphi : [0,1] \to S$ with $\varphi(0) = a$ and $\varphi(1) = b$.

Proposition: For S path connected and $a, b \in S$, $\pi_1(S, a)$ is isomorphic to $\pi_1(S, b)$.

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Let φ be a path between a and b.
The isomorphism is given by $\rho \mapsto \varphi\rho\varphi^1$.

Fundamental group of a path connected S

In light of the previous proposition, we will denote by $\pi_1(S)$ the group $\pi_1(S, *)$ for any $* \in S$.

$\pi_1(\mathbb{S}^n)$ **is trivial for** $n \geq 2$

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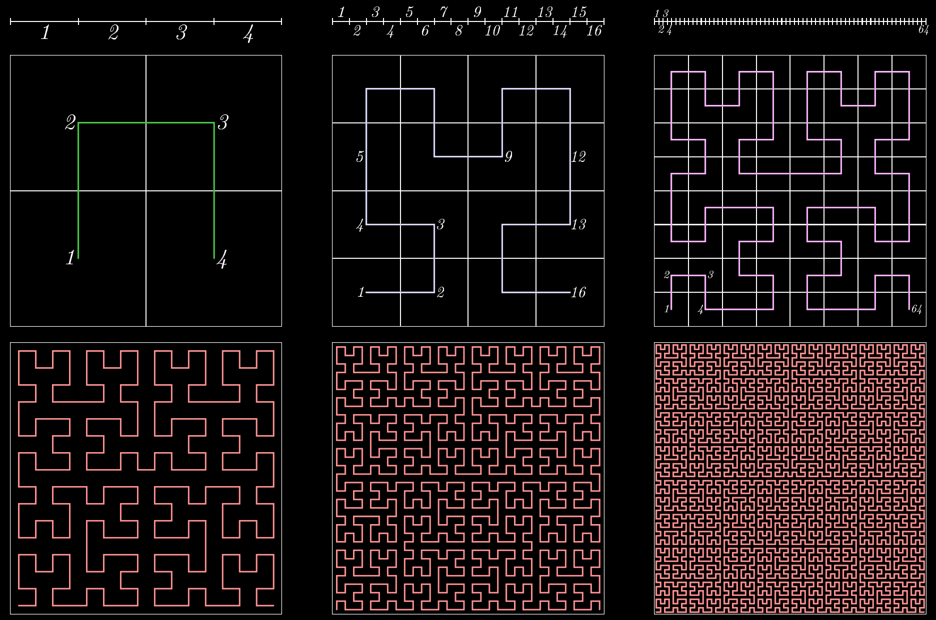
- For a loop ρ, find $x \in \mathbb{S}^2$ not in the image of ρ.
- Project stereographically from x to $\mathbb{R}^n$.
- Contract in $\mathbb{R}^n$.

The previous argument is NOT always correct

We need to carefully deform a curve ρ that fills the whole sphere to a one that doesn't in order to find the x not in the image of ρ.



Group homomorphisms

Let G be a group with multiplication $*$ and H a group with multiplication $\odot$.
A function $f : G \to H$ is called a *homomorphism* if it preserves the group stucutre, i.e. for $g, h \in G$:

$$f(g * h) = f(g) \odot f(h)$$

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Don’t confuse them with hom*e*omorphisms! It’s a standard terminology :/

π_1 is a functor = respects maps

For $* \in S$ and $f : S \to T$, there is an induced homomorphism

$$\pi_1(f) : \pi_1(S, *) \to \pi_1(T, f(*))$$

mapping the class of a loop
$\rho : [0, 1] \to S$ to the class of a loop $f \circ \rho : [0, 1] \to T$.

π_1 is a functor = respects maps + their compositions

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It respects the composition of maps, so for $f : S \to T$ and $g : T \to U$,
$\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f)$.

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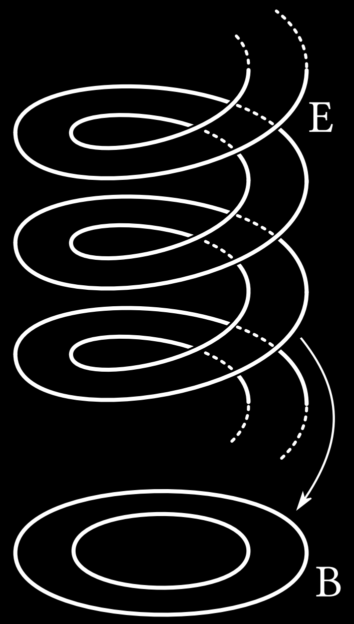
This means we can think of π_1 as a "portal" from spaces to groups.

How to compute the fundamental group?

Covering of the circle by the real line

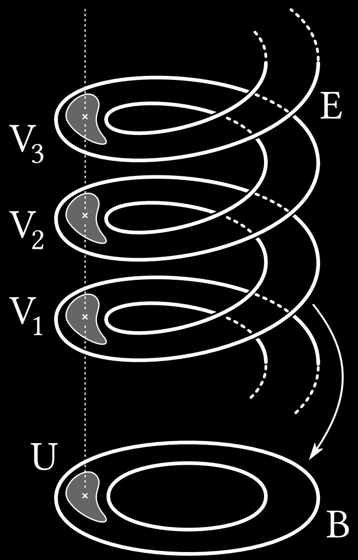
Consider the map $p : \mathbb{R}^1 \to \mathbb{S}^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$.



Coverings

A map $p : E \to B$ is called a *covering* if the following is satisfied:

- each $b \in B$ has a neighbourhood U such that $p^{-1}(U)$ is a disjoint union of spaces homeomorphic to U



Neighbourhoods formally

For a metric space M with the distance d and $x \in M$, let the open ball around x of radius r be $B(x, r) = \{y \in M : d(x, y) < r\}$.

A set $O \subset M$ is a neighbourhood of $x \in O$ if it contains some open ball around x.

Covering of the circle by the real line

The map $p : \mathbb{R}^1 \to \mathbb{S}^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$ is a covering:

For each $t \in \mathbb{R}^1$, there is an open interval $(t - \varepsilon, t + \varepsilon)$ where sin and cos are invertible.

Its preimage is a disjoint union of such intervals.

Paths in B can be uniquely lifted to E

Proposition: For a covering $E \to B$ along with:

- $b \in B$
- a path φ in B with $\varphi(0) = b$
- $e \in E$ with $p(e) = b$,

there is a unique path $\tilde{\varphi}$ in E with $\tilde{\varphi}(0) = e$ and $p(\tilde{\varphi}) = \varphi$.

The proof uses Heine-Borel theorem.

Homotopies in B can be uniquely lifted to E

Proposition: For a covering $E \to B$ along with:

- $b \in B$
- paths φ, ψ in B with $\varphi(0) = b = \psi(0)$
- $e \in E$ with $p(e) = b$,
- homotopy H between φ and ψ

there is a unique lift $\tilde{H}$ as a homotopy between $\tilde{\varphi}$ and $\tilde{\psi}$.

The proof is completely analogous to the previous.

Fibers in coverings

For a covering $p : E \to B$ and $b \in B$, the preimage $p^{-1}(b)$ is a discrete set of points.

It is called the *fiber* of b.

For a covering $p : E \to B$, a map $f : E \to E$ is called a *deck transformation* if it respects the covering, i.e. $pf = p$.

Deck transformations of a covering form a group called $G(E)$.

A covering $p : E \to B$ is called *universal* if E is path connected and $\pi_1(E)$ is trivial.

Proposition: For a universal covering $E \to B$ and $b \in B$, the deck transformations correspond to the points of the fiber $p^{-1}(b)$.

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Let $e_0 \in E$ and pick its image $f(e_0)$. For $e \in E$, pick a path φ from e_0 to e. There is a unique lift $\hat{\varphi}$ of $p\varphi$ starting at $f(e_0)$.
Define $f(e) = \hat{\varphi}(1)$. Check that this is well defined.

What is a group equivalence?

A homomorphism $f : G \to H$ is called an *isomorphism* if it has a two-sided homomorphism inverse, i.e. there is a homomorphism $g : H \to G$ such that both composites fg and gf are identity.
Then S and T are called isomorphic (just a fancy word for group-like equivalent).

Theorem: For a *universal* path connected covering $p : E \to B$, $\pi_1(B)$ is isomorphic to the group of deck transformations $G(E)$.

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Pick $b \in B$ and $e \in p^{-1}(b)$.

$[\varphi] \mapsto d_{\tilde{\varphi}(e)}$ (the deck transformation mapping e to $\tilde{\varphi}(e)$)

$d \mapsto p \circ \rho_d$ for a path ρ_d connecting e and d_e

Check that these are mutually inverse homomorphisms.

Application: the fundamental group of $\mathbb{S}^1$

We have a covering $p : \mathbb{R}^1 \to \mathbb{S}^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$. What is the group of deck transformations?

Application: the fundamental group of $\mathbb{S}^1$

We have a covering $p : \mathbb{R}^1 \to \mathbb{S}^1, p(t) = (\cos 2\pi t, \sin 2\pi t)$. What is the group of deck transformations?

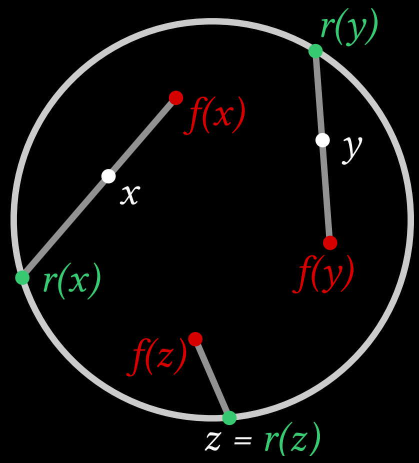
$p^{-1}((1,0)) = \mathbb{Z}$.
The deck transformations are translations mapping $\mathbb{Z}$ to $\mathbb{Z}$.
This group is isomorphic to $\mathbb{Z}$ with addition.

Application: Brouwer's fixed point theorem

Suppose there is a continuous map $\mathbb{D}^2 \to \mathbb{D}^2$ without fixed point.

Follow rays from $f(x)$ to x until they intersect the boundary circle $\mathbb{S}^1$.

This gives a continuous map $r : \mathbb{D}^2 \to \mathbb{S}^1$ which is an identity on $\mathbb{S}^1$.



Application: Brouwer's fixed point theorem

There is also the inclusion $i : \mathbb{S}^1 \to \mathbb{D}^2$ and the composite $ri : \mathbb{S}^1 \to \mathbb{D}^2 \to \mathbb{S}^1$ is identity.

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Apply the fundamental group to this sequence:
$\pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{D}^2) \to \pi_1(\mathbb{S}^1)$ must be identity.

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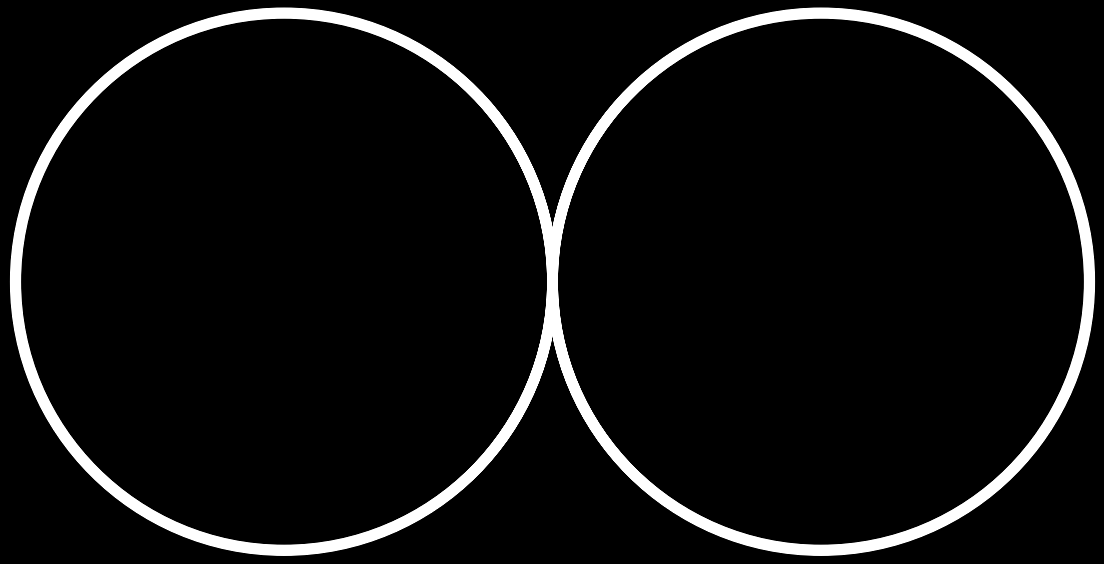
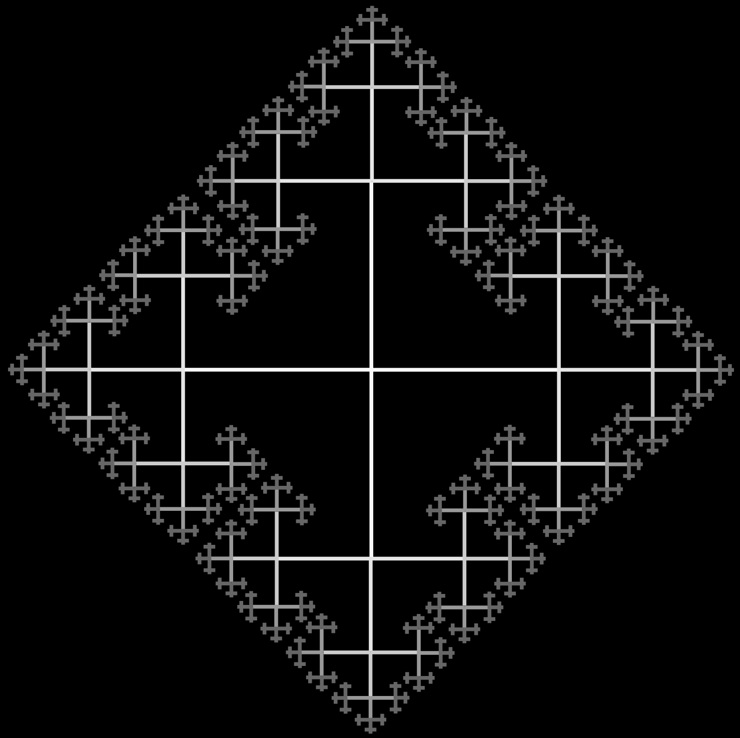
But this means we must have homomorphisms
$\mathbb{Z} \to 1 \to \mathbb{Z}$
whose composite is identity. A contradiction!

Application: wedge of 2 circles



Wedge of 2 circles: covering

An infinite tree.



Free group with 2 generators

- Elements: strings of letters a, b, a^{-1}, b^{-1}
- Group operation: concatenation of words
- Neutral element: empty word
- Substrings aa^{-1}, $b^{-1}b$ and so on get erased

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- Elements: strings of letters a, b, a^{-1}, b^{-1}
- Group operation: concatenation of words
- Neutral element: empty word
- Substrings aa^{-1}, $b^{-1}b$ and so on get erased

It's the fundamental group of the wedge of 2 circles.

Analogously, the fundamental group of the wedge of n circles is the free group with n generators.

What is the fundamental group of a connected graph?

Theorem: Let B be a space for which there exists an universal covering.

Then there is a correspondence:

coverings of B

$\leftrightarrow$

subgroups of $\pi_1(B)$

Nielsen-Schreier theorem: Every subgroup of a free group is free.

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Proof: A covering of a wedge of circles is a graph.

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Each rotation is represented by two quaternions: $q, -q$.

This shows that $\mathbb{S}^3$ (the space of unit quaternions) is a double cover of $SO(3)$. So the fundamental group of $SO(3)$ must be $\mathbb{Z}_2$.

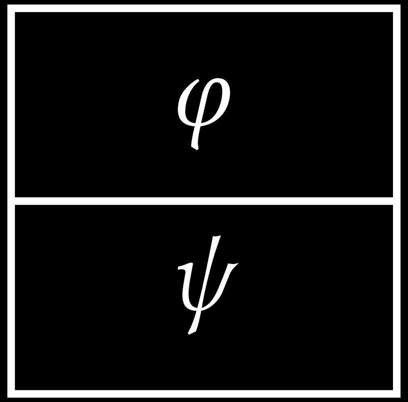
Higher homotopy groups (of a space S at $* \in S$)

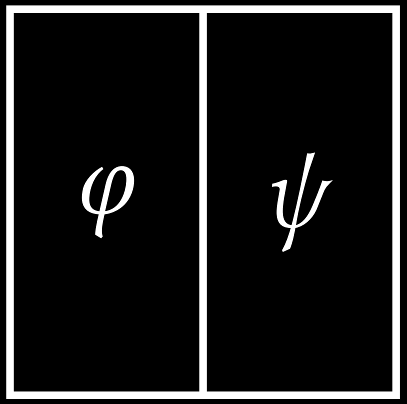
$\Omega^n(S, *) = \{$ maps $[0,1]^n \to S$ with the boundary mapped to $*\}$

$\pi_n(S, *) = \Omega^n(S, *)/\sim$ (factored by homotopies)

Group structure on $\pi_2(S, *)$

We can define horizontal and vertical composition.





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id	ψ

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These coincide and it shows the composition is commutative.

The same holds for $\pi_n(S, *)$, $n > 2$.

Theorem: A covering $p : E \to B$ provides an isomorphism $\pi_n(p) : \pi_n(E) \to \pi_n(B)$ for all $n \geq 2$.

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Thanks to this, $\pi_n(\mathbb{S}^1)$ is trivial for $n \geq 2$.

Homotopy groups of spheres: facts

- $\pi_i(\mathbb{S}^n)$ is trivial for $i < n$
- $\pi_n(\mathbb{S}^n) = \mathbb{Z}$
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- $\pi_n(\mathbb{S}^n) = \mathbb{Z}$
- $\pi_i(\mathbb{S}^n)$ is finite for $i > n$, except for $\pi_{4k-1}(\mathbb{S}^{2k+1})$ (Serre)
- $\pi_i(\mathbb{S}^n)$ is isomorphic to $\pi_{i+1}(\mathbb{S}^{n+1})$ for $i < 2n - 1$ (Freudenthal suspension theorem)

In general, the patterns of higher homotopy groups of spheres remain shrouded in mystery.