

Convention: The word *map* here means a continuous function between topological spaces.

Definition 1. A *homotopy* between maps $f, g : X \rightarrow Y$ is a map

$$H : X \times [0, 1] \rightarrow Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. If a homotopy exists, we call f and g *homotopic* and write $f \sim g$.

We denote by $[X, Y]$ the set of homotopy classes of maps between X and Y .

$$\begin{array}{ccc} & f & \\ X & \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow H \\ \xrightarrow{\quad} \end{array} & Y \\ & g & \end{array}$$

Exercise 2. Show that if $h, h' : X \rightarrow Y$ are homotopic and $k, k' : Y \rightarrow Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.

Conclude that topological spaces and homotopy classes of maps form a category. It is called the *homotopy category of topological spaces* and denoted hoTop .

Definition 3. A *homotopy equivalence* is an isomorphism in hoTop . Explicitly, it is a map $f : X \rightarrow Y$ such that there is a map $g : Y \rightarrow X$ with

$$f \circ g \sim \text{Id}, \quad g \circ f \sim \text{Id}$$

X and Y are then called *homotopy equivalent*.

A space is called *contractible* if it is homotopy equivalent to a point $*$.

Compare the preceding definition to that of *homeomorphism*, where there is the strict equality $=$ instead of $\sim$.

Exercise 4. 1. Show that $I = [0, 1]$ and $\mathbb{R}$ are contractible.

2. Show that for Y contractible and any X , $[X, Y]$ has a single element.

3. Show that a contractible space is path connected.

4. Show that if Y is path connected, $[*, Y]$ has a single element.

5. Show that for X contractible and Y path connected, $[X, Y]$ has a single element.

Definition 5. A *deformation retraction* of a subspace $A \subseteq X$ is a map

$$H : X \times [0, 1] \rightarrow X$$

such that $H(x, 0) = x$, $H(x, 1) \in A$ for all $x \in X$ and $H(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$.

Exercise 6. Construct an explicit deformation retraction of $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$.

Exercise 7. Show that if there is a deformation retraction H of $A \subseteq X$, the inclusion $i : A \hookrightarrow X$ and the map $r : X \rightarrow A \subseteq X$, $r(x) = H(x, 1)$ are homotopy equivalences.

Exercise 8. Show that there is a deformation retraction between the torus with one point removed and the graph of its meridian and latitude circles intersecting in a point.

Exercise 9. Show that if X deformation retracts into a point $x \in X$, then for every neighbourhood U of x , there is a neighbourhood $x \in V \subseteq U$ homotopy equivalent to a point.