Action of the fundamental group

Let $p : (\tilde{X}, \tilde{x}) \to (X, x)$ be a covering space with $\tilde{X}, X$ are path-connected. We construct a map $\cdot : p^{-1}(x) \times \pi_1(X, x) \to p^{-1}(x)$. Given $y \in p^{-1}(x)$ and $[\gamma] \in \pi_1(X, x)$, put $y \cdot [\gamma] = \tilde{\gamma}_y(1)$, where $\tilde{\gamma}_y$ is the lift of γ at y. This definition does not depend of the choice of the lift. Why?

Exercise 1.

- Show that this defines a right action of $\pi_1(X, x)$ on $p^{-1}(x)$, i.e., $y \cdot 1 = y$ and $(y \cdot [\alpha]) \cdot [\beta] = y \cdot ([\alpha][\beta])$. Show that this action is transitive, i.e., for each $y, z \in p^{-1}(x)$, there exists $[\gamma] \in \pi_1(X, x)$ s.t. $y \cdot [\gamma] = z$.
- For the base point $\tilde{x}$, consider the stabilizer $S = \{[\gamma] \in \pi_1(X, x) \mid \tilde{x} \cdot [\gamma] = \tilde{x}\}$. Observe that $S = p_*(\pi_1(\tilde{X}, \tilde{x})) \subseteq \pi_1(X, x)$. This follows directly from the proposition that p_* is injective.
- Show that the map $S[\gamma] \mapsto \tilde{x} \cdot [\gamma]$ is a bijection between the set of right cosets of S and the fiber $p^{-1}(x)$.

We have therefore proven the following.

Theorem 2. Let $p : (\tilde{X}, \tilde{x}) \to (X, x)$ be a covering space with $\tilde{X}, \tilde{X}$ path-connected. Then there is a bijection between the set of right cosets of $p_*\pi_1(\tilde{X}, \tilde{x})$ in $\pi_1(X, x)$ and the fiber $p^{-1}(x)$.

In other words, the number of sheets of the covering is equal to the index of the subgroup of the fundamental group that corresponds to the said covering. Refine this statement for the case of universal covering.

Exercise 3. $\mathbb{R}P^n$ has a universal double-covering $\mathbb{S}^n$. Compute its fundamental group.

Subgroups of the free group

Define a **graph** to be a 1-dimensional CW complex, in other words a space obtained by attaching to a discrete space X_0 ('vertices') a collection of 1-cells e_i ('edges'). A **subgraph** of X is a subspace Y that is a union of vertices and edges of X such that $e_i \subset Y$ implies $\bar{e}_i \subset Y$ (if an edge is in the subgraph, then its vertices are as well). A **tree** is a contractible graph. We call a tree in X **maximal** if it contains all the vertices of X.

Exercise 4. Nielsen–Schreier theorem states that every subgroup of a free group is free. Prove it using coverings of graphs.

- Show that every connected graph contains a maximal tree, and in fact any tree in the graph is contained in some maximal tree.
- Show that for a connected graph X with maximal tree T, $\pi_1(X)$ is a free group with basis the classes f_α corresponding to the edges e_α of $X - T$.
- Show that every covering space of a graph is also a graph, with vertices and edges the lifts of the vertices and edges in the base graph.
- Realize the free group as a fundamental group of a wedge of circles and deduce the theorem.