Theorem (Seifert-van Kampen)**.** Let $X = U \cup V$ for U, V open and $U, V, U \cap V$ path-connected, $x \in U, V$. Then the following diagram is a pushout in the category of groups:

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \longrightarrow & \pi_1(U, x) \\ \downarrow & & \downarrow \\ \pi_1(V, x) & \longrightarrow & \pi_1(X, x) \end{array}$$

Definition 1. A map $p : E \to B$ is called a *covering* if for every $b \in B$, there is a neighbourhood $b \in U \subseteq B$ such that $p^{-1}(U)$ is homeomorphic to a disjoint union of a copies of U. It is called *universal* if E is path-connected and $\pi_1(E)$ is trivial.

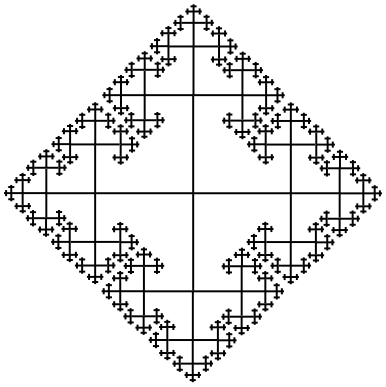
Proposition 2. For a universal covering $p : E \to B$ and $x \in B$, $\pi_1(B) \simeq p^{-1}(x)$ as sets. $\pi_1(B)$ is isomorphic to the group of *deck transformations* of p: maps $f : E \to E$ such that $p \circ f = p$.

Exercise 1. Compute $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ in two ways.

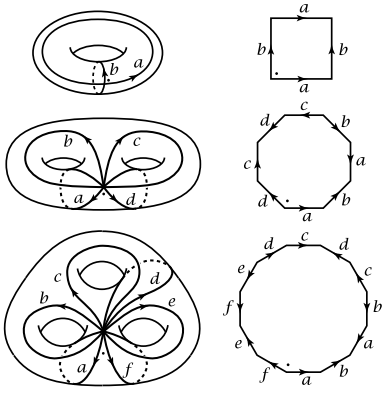
1. Using the van Kampen's theorem (with U, V small neighbourhoods of both circles).
2. Show that the universal covering is given by the infinite fractal tree in the Figure (a), with vertical line segments covering one circle and horizontal ones covering the other.
3. Generalize the statement to $\vee_n \mathbb{S}^1$.

Definition 3. For $g \geq 1$, a genus g surface M_g (or "g-fold donut") is a surface obtained from $4g$-gon with sides subsequently labeled by $a_1, b_1, a_1, b_1, \ldots, a_g, b_g, a_g, b_g$ by gluing the sides of the same label with the opposite orientation. This $4g$-gon is called its *fundamental polygon*.

Exercise 2. Using the van Kampen theorem, compute the $\pi_1(M_g)$. (*Hint:* Pick U the interior of the disk in the centre of the fundamental polygon and V the complement of the centre of the fundamental polygon in M_g.)
Show that the universal covering of the torus is $\mathbb{R}^2$. What about the higher genus surfaces?



(a) The universal covering of $\mathbb{S}^1 \wedge \mathbb{S}^1$



(b) Genus g surface, Hatcher

Exercise 3. Using the van Kampen's theorem, compute $\pi_1(\mathbb{S}^2)$. (*Hint:* Pick as U a neighbourhood of the northern and as V a neighourhood of the southern hemisphere.)

Exercise 4. Analogously to torus, compute the fundamental group of the projective plane from its fundamental square. Show that $\mathbb{S}^2$ is its universal covering.

Exercise 5. Recall that a rotation in $\mathbb{R}^3$ can be represented by a multiplication $v \mapsto qvq^{-1}$ for a suitable unit quaternion q (where v is considered as a purely imaginary quaternion). In particular, q and $-q$ represent the same rotation. Show that this defines a double covering $\mathbb{S}^3 \to SO(3)$ (meaning every point has two preimages). Conclude that $\pi_1(SO(3)) \simeq \mathbb{Z}_2$.

Definition 4. Let ϕ be a loop on M_g and B a small band around it homeomorphic to the annulus $\mathbb{S}^1 \times I$, with the coordinates (s, t). The *Dehn twist* around ϕ is a map $f : M_g \to M_g$, which is the identity outside B and on B defined by

$$f(s, t) = (se^{i2\pi t}, t)$$

Exercise 6 (Bonus)**.** Play 30 levels of transforming loops on M_2 with Dehn twists.

http://www.aharalab.sakura.ne.jp/teruaki.html