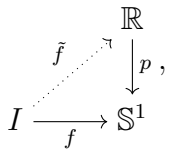
Theorem 1. $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$. A representanive corresponding to $n \in \mathbb{Z}$ is given by winding around the circle n times in the direction determined by the sign of n.

Idea. Consider the covering



where $p(r) = (\cos 2\pi r, \sin 2\pi r)$. For each path $f : I \to \mathbb{S}^1$ starting at $* \in \mathbb{S}^1$ and each $x \in p^{-1}(*)$, there exists a unique lift $\tilde{f} : I \to \mathbb{R}$ s.t. $p \circ \tilde{f} = f$. Similarly, there exist unique lifts of homotopies of paths.
Fix a loop $f : I \to \mathbb{S}^1$, it has a unique lift $\tilde{f}$ starting at 0. Moreover $\tilde{f}(1) = n$, for some $n \in \mathbb{Z}$. The path $\tilde{f}$ is homotopic to the path $\tilde{\omega}_n : s \mapsto ns$. This produces a homotopy of f to a loop ω_n that winds n times around $\mathbb{S}^1$. The number n is uniquely determined by f. ■

Exercise 2 (Fundamental theorem of algebra)**.** Prove that every non-constant complex polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

has a root (z such that $p(z) = 0$).

- For each real $r \geq 0$, the formula

  $$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}$$

  defines a loop $f_r : \mathbb{S}^1 \to \mathbb{S}^1 \subset \mathbb{C}$ based at 1. Show that each f_r is nullhomotopic, i.e. $[f_r] \in \pi_1(\mathbb{S})$ is zero.

- Given a large r, bigger than 1 and $|a_1| + \cdots + |a_n|$, show that

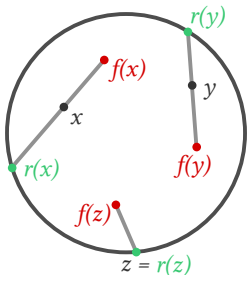
  $$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

  has no roots on the circle $|z| = r$, for $0 \leq t \leq r$.

- Replace p by p_t in the formula f_r and vary t from 1 to 0. This produces *some* homotopy. Conclude the Fundamental theorem of algebra.

Exercise 3 (Brouwer's fixed point theorem)**.** Prove that you can never perfectly mix coffee: every continuous map $f : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point $x = f(x)$.

1. Suppose f has no fixed point. For every x, draw a ray from $f(x)$ to x and mark by $r(x)$ the point of its intersection with the boundary circle $\mathbb{S}^1$. The function $x \mapsto r(x)$ is continuous.



2. Show that the composition $\mathbb{S}^1 \xrightarrow{i} \mathbb{D}^2 \xrightarrow{r} \mathbb{S}^1$ is the identity, where i is the inclusion of the boundary circle.

3. Find a contradiction by applying π_1 to the preceding composition.

Exercise 4 (Borsuk-Ulam theorem)**.** Prove that at all times, there is a pair of antipodal places on the Earth with exactly the same temperature and atmospheric pressure: For every continuous map $f : \mathbb{S}^2 \to \mathbb{R}^2$, there is $x \in \mathbb{S}^2$ such that $f(x) = f(-x)$.



- Define $g : \mathbb{S}^2 \to \mathbb{S}^1$ and $\eta : \mathbb{S}^1 \to \mathbb{S}^2 \subset \mathbb{R}^3$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \quad \text{and} \quad \eta(s) = (\cos 2\pi s, \sin 2\pi s, 0).$$

  And denote their composition $h = g \circ \eta$.

- $g(-x) = -g(x)$ implies that $h(s+\frac{1}{2}) = -h(s)$, for $s \in [0, \frac{1}{2}]$. Consider the lift $\tilde{h} : I \to \mathbb{R}$ of h. Show that $\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$, where q is an odd integer.

- Show that q does not depend on s, i.e., q is constant. Conclude that h is a q-fold loop in $\mathbb{S}^1$. Since q is odd, h is non-trivial.

- On the other hand, show that η is null-homotopic in $\mathbb{S}^2$, derive the contradiction.

Exercise 5. Using Borsuk-Ulam theorem, show that if $\mathbb{S}^2$ is expressed as a union of three closed sets A_1, A_2, A_3, then at least one of them contains a pair of antipodal points $\{x, -x\}$. *Hint:* Consider the function (d_1, d_2) with $d_i(x) = \inf_{y \in A_i} |x - y|$.

Exercise 6. Does Borsuk-Ulam theorem hold for the torus? Must there for every $f : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}^2$ be a point $(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1$ such that $f(x, y) = f(-x, -y)$.