**Definition 1.** Let $I$ be the interval $[0,1]$. A *path* is a continous map from $I$.
For paths $\phi, \psi : I \to X$ with $\phi(1) = \psi(0)$, their composition is defined by concatenation

$$\phi \cdot \psi(t) := \begin{cases} \phi(2t) & t \in [0, \frac{1}{2}] \\ \psi(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

For a path $\phi$, its inverse path is given by $\phi^{-1}(t) = \phi(1-t)$.
A *path homotopy* is a homotopy between paths not moving the endpoints. $\phi \simeq \psi$ here means there is a path homotopy between $\phi$ and $\psi$.

**Exercise 1.** Show that the composition of paths satisfies the following cancellation property: if $\phi \cdot \psi \simeq \phi' \cdot \psi'$ and $\psi \simeq \psi'$, then $\phi \simeq \phi'$.

**Exercise 2.** Show for paths $\phi, \psi, \lambda$, if the compositions $\phi \cdot \psi$ and $\psi \cdot \lambda$, are defined, the paths $(\phi \cdot \psi) \cdot \lambda$ and $\phi \cdot (\psi \cdot \lambda)$ are not in general equal, since they have different parametrisations. Construct a path homotopy $A$ between them (it is called an *associator* since it witnesses associativity up to homotopy).
*Bonus:* How many ways are there to bracket the composition of 4 paths $\phi_1 \cdot \phi_2 \cdot \phi_3 \cdot \phi_4$? What about $n$ paths? In how many ways can we use the associator homotopy $A$ to construct a homotopy from $((\phi_1 \cdot \phi_2) \cdot \phi_3) \cdot \phi_4$ to $\phi_1 \cdot (\phi_2 \cdot (\phi_3 \cdot \phi_4))$?

**Definition 2.** For a space $X$, $x \in X$, the loop space of $X$ based at $x$ is the set

$$\Omega(X, x) := \{\phi : I \to X : \phi(0) = x = \phi(1)\}$$

Equivalently, it is the set of pointed maps from $\mathbb{S}^1$ to $(X, x)$.
The *fundamental group* is the set $\pi_1(X, x) = \Omega(X, x)/\simeq$ of its path homotopy classes, with the group structure induced by the path composition and inverse.
A path $\tau$ with $\tau(0) = x$, $\tau(1) = y$ induces a *change of basepoint* map

$$\begin{aligned} \pi_1(X, x) &\to \pi_1(X, y) \\ [\phi] &\mapsto [\tau \cdot \phi \cdot \tau^{-1}] \end{aligned}$$

**Exercise 3.** Show that the set $\pi_1(X, x)$ is in bijection with the set of homotopy classes of maps from $\mathbb{S}^1$ to $X$ mapping a chosen $* \in \mathbb{S}^1$ to $x$.

**Exercise 4.** Show that the change of basepoint map is a group isomorphism by finding its inverse. Show that it only depends on the homotopy class of $\tau$.

**Exercise 5.** Show that for $X$ path connected, $\pi_1(X, x)$ is abelian if and only if for any $y \in X$, there is a unique change of basepoint map $\pi_1(X, x) \to \pi_1(X, y)$.

**Exercise 6.** Let $[\mathbb{S}^1, X]$ be the set of maps $\mathbb{S}^1 \to X$ up to homotopy, without any condition of basepoints. There is a natural projection $\Pi : \pi_1(X, x) \to [\mathbb{S}^1, X]$. Suppose $X$ is path-connected. Show that $\Pi$ is onto and $\Pi(\phi) = \Pi(\psi)$ if and only if $\phi$ and $\psi$ are conjugate in $\pi_1(X, x)$ (i.e. there is $\lambda$ s.t. $\phi = \lambda^{-1} \cdot \psi \cdot \lambda$).
Hence $\Pi$ is a bijection between $[\mathbb{S}^1, X]$ and the conjugacy classes of $\pi_1(X, x)$.

## Suspension-loopspace adjunction

For topological spaces $X$, $Y$, we can topologize the set $X^Y$ of continuous maps $Y \to X$ by giving it the *compact-open topology*: its subbase consists of the sets $V(K,U) = \{f : Y \to X : f(K) \subseteq U\}$ for $K \subseteq Y$ compact and $U \subseteq X$ open. This means the open sets of $X^Y$ are unions of finite intersections of these sets.
The only property of this topology that we will need is the following: for any $I$ compact, there is a natural bijection of sets

$$\mathrm{Hom}_{\mathrm{Top}}(X \times I, Y) \simeq \mathrm{Hom}_{\mathrm{Top}}(X, Y^I)$$

where $\mathrm{Hom}_{\mathrm{Top}}(-,-)$ is the set of continuous maps between spaces, given by the so called *currying adjunction* mapping $f(-,-)$ to $x \mapsto (z \mapsto f(x,z))$.

**Exercise 7.** Show that for pointed spaces $(X,x)$, $(Y,y)$, this restricts to a natural bijection

$$\mathrm{Hom}_{\mathrm{Top}_*}(X \wedge \mathbb{S}^1, Y) \simeq \mathrm{Hom}_{\mathrm{Top}_*}(X, \Omega Y)$$

where $\mathrm{Hom}_{\mathrm{Top}_*}(-,-)$ is the set of continuous maps between spaces preserving the basepoints, $\wedge$ is the smash product $X \wedge Y = (X \times Y)/(\{*_X\} \times Y \sim X \times \{*_Y\})$. $\Omega(X,x)$ can be equivalently considered as the set of pointed maps from $\mathbb{S}^1$ to $(X,x)$, with the subspace topology and the constant loop as a basepoint.

**Exercise 8.** We call $X \wedge \mathbb{S}^1$ the *suspension* of $X$. Show that it is homeomorphic to the quotient space

$$\Sigma X = X \times I/(X \times \{0\} \sqcup X \times \{1\} \sqcup \{x\} \times I).$$

Show that for $n \geq 0$, $\Sigma\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$.

**Exercise 9.** Let $\pi_0(X)$ be the set of path components of $X$. Conclude that there is a bijection of sets $\pi_1(X,x) \simeq \pi_0(\Omega(X,x))$.

This show that $\Omega$ is the correct notion of a loop space, with points corresponding to loops and paths to homotopies of loops.

**Exercise 10.** Define $\pi_n(X,x)$ as the set of pointed maps from $\mathbb{S}^n$ to $(X,x)$, quotiented by homotopies preserving the basepoint. Show that there is a bijection $\pi_n(X,x) \simeq \pi_{n-1}(\Omega(X,x))$.

One of the (equivalent) ways of equipping $\pi_n(X,x)$ with a group structure is then by composition in $\Omega(X,x)$.