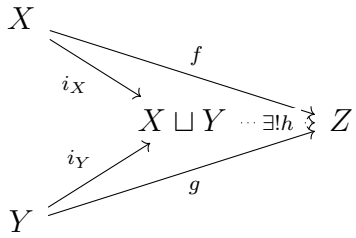
Definition 1. A *coproduct* of topological spaces X, Y is a space $X \sqcup Y$, along with maps $i_X : X \to X \sqcup Y$, $i_Y : Y \to X \sqcup Y$ satisfying the following *universal property*: for any topological space Z and maps $f : X \to Z$, $g : Y \to Z$, there exists a unique map $h : X \sqcup Y \to Z$ making the following diagram commute.



The diagram commuting means that all compositions of depicted functions from some source to some target are equal. In this case $h \circ i_X = f$ and $h \circ i_Y = g$.

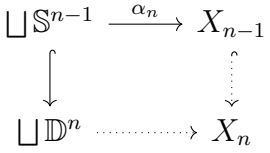
Exercise 1. Show that the disjoint union of topological spaces (along with their inclusions) is their coproduct. Hence the abuse of notation here is slight.

Exercise 2. Show that the coproduct is unique up to isomorphism (if it exists).

Exercise 3. Does the disjoint union give the coproduct in the category of **pointed** topological spaces?

Definition 2. We define the notion of a CW complex iteratively.

1. A 0-dimensional CW complex is a discrete set of points.
2. An n-dimensional CW complex, $n > 0$, is a space X_n obtained by attaching a collection of disks $\bigsqcup \mathbb{D}^n$ along their boundary spheres $\bigsqcup \mathbb{S}^{n-1}$ to an $(n-1)$-dimensional CW complex X_{n-1}.



   This means X_n is the quotient space $(X_{n-1} \sqcup \bigsqcup \mathbb{D}^n)/\sim$, where the equivalence relation identifies $x \in \bigsqcup \mathbb{S}^{n-1} \subset \bigsqcup \mathbb{D}^n$ with it image under the attaching map α_n.

3. For a sequence of inclusions obtained by the process from the step 2

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \hookrightarrow X_n \hookrightarrow \dots$$

   a CW complex X is the union $\bigcup_n X_n$.

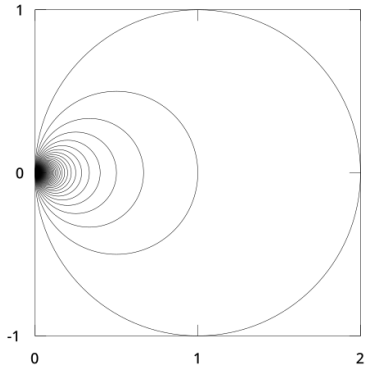
Proposition 3. For a CW complex X and a contractible subcomplex $A \subseteq X$, the quotient map $X \to X/A$ is a homotopy equivalence.

Exercise 4. Show that the space obtained from $\mathbb{S}^2$ by attaching n copies of $\mathbb{D}^2$ along any collection of n circles in $\mathbb{S}^2$ is homotopy equivalent to the wedge sum of $n+1$ 2-spheres.

Exercise 5. Show that the subspace $K \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle is homotopy equivalent to $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$.

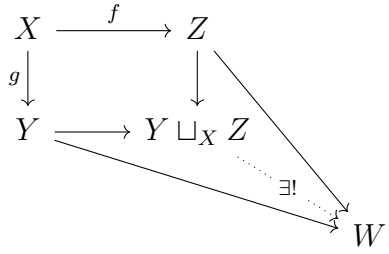
Proposition 4. Every compact *(i.e., every open cover has a finite subcover)* subset of a CW complex lies in a finite subcomplex.

Exercise 6. The **Hawaiian earring** $\mathbb{H}$ is the topological space defined by the union of circles in $\mathbb{R}^2$ with center $(\frac{1}{n}, 0)$ and radius $\frac{1}{n}$, for each $n \in \mathbb{N}$.



The space $\mathbb{H}$ is compact, therefore it is not homeomorphic to an infinite wedge of circles. Can $\mathbb{H}$ be given *any* structure of a CW complex?

Definition 5. For topological spaces X, Y, Z and maps $f : X \to Y$, $g : Y \to Z$, their *pushout* is a space $Y \sqcup_X Z$ with maps $Y \to Y \sqcup_X Z$, $Y \to Y \sqcup_X Z$ satisfying the following universal property: For any space W and maps $X \to W$ and $Y \to W$ making the following diagram commute



there is a unique map $Y \sqcup_X Z \to W$ making the diagram commute.

Exercise 7. Show that $Y \sqcup_X Z$ is given by the quotient space $Y \sqcup Z / \sim$, where $f(x) \sim g(x)$ for all $x \in X$. Show that pushouts are unique up to isomorphism.

Remark. This shows gluings can be represented as pushouts. In particular, CW complexes are iterated pushouts of disks along their boundary spheres.