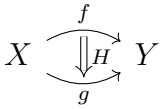
Convention: The word *map* here means a continuous function between topological spaces.

Definition 1. A *homotopy* between maps $f, g : X \to Y$ is a map

$$H : X \times [0,1] \to Y$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If a homotopy exists, we call f and g *homotopic* and write $f \sim g$.
We denote by $[X, Y]$ the set of homotopy classes of maps between X and Y.



Exercise 1. Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.
Conclude that topological spaces and homotopy classes of maps form a category. It is called the *homotopy category of topological spaces* and denoted hoTop.

Definition 2. A *homotopy equivalence* is an isomorphism in hoTop. Explicitly, it is a map $f : X \to Y$ such that there is a map $g : Y \to X$ with

$$f \circ g \sim \mathrm{Id},\ g \circ f \sim \mathrm{Id}$$

X and Y are then called *homotopy equivalent*.
A space is called *contractible* if it is homotopy equivalent to a point $*$.

Compare the preceding definition to that of *homeomorphism*, where there is the strict equality $=$ instead of $\sim$.

Exercise 2.
1. Show that $I = [0,1]$ and $\mathbb{R}$ are contractible.
2. Show that for Y contractible and any X, $[X, Y]$ has a single element.
3. Show that a contractible space is path connected.
4. Show that if Y is path connected, $[*, Y]$ has a single element.
5. Show that for X contractible and Y path connected, $[X, Y]$ has a single element.

Definition 3. A *deformation retraction* of a subspace $A \subseteq X$ is a map

$$H : X \times [0,1] \to X$$

such that $H(x,0) = x$, $H(x,1) \in A$ for all $x \in X$ and $H(a,t) = a$ for all $a \in A$ and $t \in [0,1]$.

Exercise 3. Construct an explicit deformation retraction of $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n \backslash \{0\}$.

Exercise 4. Show that if there is a deformation retraction H of $A \subseteq X$, the inclusion $i : A \hookrightarrow X$ and the map $r : X \to A \subseteq X$, $r(x) = H(x,1)$ are homotopy equivalences.

Exercise 5. Show that there is a deformation retraction between the torus with one point removed and the graph of its meridian and lattitude circles intersecting in a point.

Exercise 6. Show that if X deformation retracts onto a point $x \in X$, then for every neighbourhood U of x, there is a neighbourhood $x \in V \subseteq U$ with the inclusion $V \hookrightarrow U$ nullhomotopic.