Lecture 6
Classification of covering spaces and deck transformations

Last time we associated to a covering a subgroup of $\pi_1(X, x_0)$: the classes of loops whose lifts from the chosen point upstairs close. How much does this subgroup tell us? We will see that it determines the whole covering, provided the covering space is path connected.

Let X be path connected, locally path connected and semilocally simply connected, and fix $x_0 \in X$. Write $G = \pi_1(X, x_0)$. We consider nonempty connected covering spaces of X. They are locally path connected, since locally they are homeomorphic to open subsets of X, and hence they are also path connected.

1 When are two coverings the same?

It is not enough for the spaces upstairs to be homeomorphic. For example, $z \mapsto z^2$ and $z \mapsto z^3$ are two different coverings $S^1 \to S^1$.

Definition 6.1. An **isomorphism of coverings** $p_1 : \widetilde{X}_1 \to X$ and $p_2 : \widetilde{X}_2 \to X$ is a homeomorphism $F : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 F = p_1$:

$$\begin{array}{ccc} \widetilde{X}_1 & \xrightarrow{\quad F \quad} & \widetilde{X}_2 \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

If points $\widetilde{x}_1 \in p_1^{-1}(x_0)$ and $\widetilde{x}_2 \in p_2^{-1}(x_0)$ are chosen, the isomorphism is **based** if $F(\widetilde{x}_1) = \widetilde{x}_2$.

Thus an isomorphism sends each point of $\widetilde{X}_1$ to a point of $\widetilde{X}_2$ above the same point of X. A based isomorphism must also respect the choice of the starting point for lifting loops.

Proposition 6.2. *Two based connected coverings admit a based isomorphism if and only if their associated subgroups*

$$H_i = (p_i)_* \pi_1(\widetilde{X}_i, \widetilde{x}_i) \subseteq G \qquad (i = 1, 2)$$

are equal, $H_1 = H_2$. The based isomorphism, when it exists, is unique.

Proof. Since F is a based homeomorphis, then F_* is an isomorphism and $F_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = \pi_1(\widetilde{X}_2, \widetilde{x}_2)$. From the commutativity we get $(p_1)_* = (p_2)_* F_*$, so

$$H_1 = (p_1)_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_* F_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = (p_2)_* \pi_1(\widetilde{X}_2, \widetilde{x}_2) = H_2.$$

Conversely, suppose $H_1 = H_2$. We want to construct the arrow F in the diagram. This is precisely a lift of p_1 through p_2, with prescribed value $F(\widetilde{x}_1) = \widetilde{x}_2$:

$$\begin{array}{ccc} & & \widetilde{X_2} \\ & {\scriptstyle F}\nearrow & \big\downarrow {\scriptstyle p_2} \\ \widetilde{X_1} & \xrightarrow[p_1]{} & X \end{array}$$

From the proposition at the end of Lecture 5 we know that such a lift F exists if and only if $H_1 \subseteq H_2$, moreover, F is unique. Similarly, $H_2 \subseteq H_1$ gives a map $K : \widetilde{X}_2 \to \widetilde{X}_1$ with $p_1 K = p_2$ and $K(\widetilde{x}_2) = \widetilde{x}_1$.

We still have to check that these maps are inverse to each other to prove that F is a homeomorphism. Both KF and $\mathrm{id}_{\widetilde{X}_1}$ are lifts of p_1 through p_1, and both take $\widetilde{x}_1$ to itself. Uniqueness of lifts gives $KF = \mathrm{id}_{\widetilde{X}_1}$. In the same way, $FK = \mathrm{id}_{\widetilde{X}_2}$. □

2 Constructing a covering from a subgroup

We now ask the converse question: given a subgroup $H \subseteq G$, can we find a covering whose associated subgroup is H? The universal cover construction from Lecture 5 is a useful starting point which solves the problem for the trivial H.

Let $q : \widehat{X} \to X$ be that universal cover. A point of $\widehat{X}$ is a class $[\gamma]$ of paths starting at x_0. The projection map is $q([\gamma]) = \gamma(1)$. Define a relation on $\widehat{X}$ by

$$[\gamma] \sim_H [\gamma'] \quad \iff \quad \gamma(1) = \gamma'(1) \text{ and } [\gamma\overline{\gamma'}] \in H.$$

Here $\gamma\overline{\gamma'}$ is a loop in X at x_0. For the trivial $H = \{1\}$ the condition says exactly that γ and γ' are homotopic with endpoints fixed. For a general H, we also identify them when this loop represents a nontrivial element of H.

The relation is an equivalence relation because H is a subgroup. Let $X_H := \widehat{X}/\sim_H$ be the quotient space, with the quotient topology. Denote the point represented by $[\gamma]$ by $[\gamma]_H$, and set

$$p_H : X_H \longrightarrow X, \qquad p_H([\gamma]_H) = \gamma(1).$$

This is well defined, remember $\gamma(1) = \gamma'(1)$ from the definition of $\sim_H$. Also, X_H is path connected, because it is the image of a continuous map (the quotient map) from a path connected space $\widehat{X}$.

The map p_H is a covering map. We are not going to fully check that. Informally you can see it as follows. Let $U \subset X$ be an even neighborhood of the universal cover and $\widehat{U} \subset \widehat{X}$ be one of its sheets. No two points of $\widehat{U}$ are identified by $\sim_H$ since they all are projected to different points in X and hence cannot satisfy $\gamma(1) = \gamma'(1)$. So, $\sim_H$ can only identify entire sheets over U.

Denote $\widetilde{x}_H := [c_{x_0}]_H$, where c_{x_0} is the constant loop at x_0, this is the natural base point of X_H. Let us now check that the subgroup associated to $p_H : X_H \to X$ is H.

We already know that this subgroup are the loops in X which lift to closed loops in X_H. For a loop α at x_0, its lift to $\widehat{X}$ runs from $[c_{x_0}]$ to $[\alpha]$. Projecting this path to X_H gives the lift to X_H, running from $[c_{x_0}]_H$ to $[\alpha]_H$. It closes precisely when

$$[\alpha]_H = [c_{x_0}]_H, \quad \text{that is, precisely when } [\alpha] \in H.$$

Together with Proposition 6.2, this proves the based classification theorem.

Theorem 6.3 (Classification with a basepoint)**.** *There is a bijection*

$$\left\{ \begin{array}{c} \textit{based isomorphism classes of} \\ \textit{connected coverings of } (X, x_0) \end{array} \right\} \longleftrightarrow \{\textit{subgroups } H \subseteq \pi_1(X, x_0)\},$$

sending $(p : \widetilde{X} \to X, \widetilde{x}_0)$ *to* $p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$.

In particular, the trivial subgroup gives the universal cover, and the whole group G gives the identity covering $X \to X$.

3 Fibers and changing the basepoint

The construction also tells us how many sheets a covering has. Recall that the **right coset** of H containing $g \in G$ is the set $Hg = \{hg : h \in H\}$. In X_H, points above x_0 are represented by loops at x_0. If $g = [\gamma]$ and $g' = [\gamma']$ are their classes in G, then

$$[\gamma]_H = [\gamma']_H \quad \iff \quad gg'^{-1} \in H \quad \iff \quad Hg = Hg'.$$

Thus there is a bijection between the fiber over x_0 and the right cosets of H. The number of these cosets is called the **index** $[G : H]$; it may be infinite. These cosets need not form a quotient group: we have not assumed that H is normal.

Corollary 6.4. *The number of sheets of a connected covering with associated subgroup H is $[G : H]$.*

Now keep the covering $p : \widetilde{X} \to X$ fixed, but choose another base point $\widetilde{x}_1 \in p^{-1}(x_0)$. Does the associated subgroup change? Choose a path $\widetilde{\gamma}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$. Its projection $\gamma = p\widetilde{\gamma}$ is a loop at x_0; put $g = [\gamma]$. Write

$$H_0 = p_*\pi_1(\widetilde{X}, \widetilde{x}_0), \qquad H_1 = p_*\pi_1(\widetilde{X}, \widetilde{x}_1).$$

To turn a loop $\widetilde{\alpha}$ based at $\widetilde{x}_1$ into a loop based at $\widetilde{x}_0$, we first follow $\widetilde{\gamma}$, then $\widetilde{\alpha}$, and then $\overline{\widetilde{\gamma}}$. The homotopy class of its projection is $g[p\widetilde{\alpha}]g^{-1}$. Hence $gH_1g^{-1} \subseteq H_0$. Since the situation is symmetric, we can use the reverse path $\overline{\widetilde{\gamma}}$ to get the opposite inclusion. So

$$H_1 = g^{-1}H_0g. \tag{1}$$

We proved that switching the base point changes the associated subgroup to its conjugacy class. Can any conjugacy class be obtained in this way? Yes. Suppose we want to get $g^{-1}H_0g$. Simply take a loop at x_0 representing g, lift this loop starting at $\widetilde{x}_0$ and take the endpoint of this lift as the new base point $\widetilde{x}_1$. Then (1) will hold and the new associated subgroup H_1 will be precisely $g^{-1}H_0g$.

Thus changing the point above x_0 produces exactly the **conjugate subgroups** $g^{-1}H_0g$. The collection of all these subgroups is the **conjugacy class** of H_0.

Together with Theorem 6.3 this gives the following:

Theorem 6.5 (Classification without a basepoint)**.** *Isomorphism classes of connected coverings of X correspond bijectively to conjugacy classes of subgroups of G.*

4 Deck transformations

We have classified coverings up to isomorphism. What about self isomorphisms of a covering?

Definition 6.6. A **deck transformation** of $p : \widetilde{X} \to X$ is a homeomorphism $D : \widetilde{X} \to \widetilde{X}$ such that $pD = p$. These homeomorphisms form a group under composition, denoted $\mathrm{Deck}(p)$, the **deck transformation group**.

The identity, the composite of two such maps, and the inverse of such a map all preserve p, so they do form a group. For example, $u \mapsto u + 1$ is a deck transformation of $p : \mathbb{R} \to S^1$, since $e^{2\pi i(u+1)} = e^{2\pi iu}$. If D is a deck transformation then $\widetilde{x}$ and $D\widetilde{x}$ must lie in the same fiber for all $\widetilde{x} \in \widetilde{X}$, so D can only permute points within the same fiber. It turns out, that D is completely determined by how it acts within a single fiber; moreover, by how it moves a single point:

Proposition 6.7. *Two deck transformations agreeing at one point agree everywhere. In particular, a deck transformation which fixes a point is the identity.*

Proof. Suppose $D(\widetilde{x}_0) = E(\widetilde{x}_0)$, and let $\widetilde{y} \in \widetilde{X}$ be any other point. Choose a path $\widetilde{\alpha}$ from $\widetilde{x}_0$ to $\widetilde{y}$. The paths $D\widetilde{\alpha}$ and $E\widetilde{\alpha}$ both lift the path $p\widetilde{\alpha}$ in X. The also have the same starting point, $D(\widetilde{x}_0) = E(\widetilde{x}_0)$. By uniqueness of path lifting they are equal, so their endpoints $D(\widetilde{y})$ and $E(\widetilde{y})$ are equal. □

Fix a base point $\widetilde{x}_0 \in \widetilde{X}$ projecting to $x_0 = p(\widetilde{x}_0)$. Which points of $p^{-1}(x_0)$ can be the image of $\widetilde{x}_0$ under a deck transformation?

Let $H = p_*\pi_1(\widetilde{X}, \widetilde{x}_0)$ be the associated subgroup, and let $\widetilde{x}_1$ be the endpoint of the lift of a loop in X representing $g \in G$. By (1), the subgroup associated to the covering $(\widetilde{X}, \widetilde{x}_1)$ is $g^{-1}Hg$. A deck transformation taking $\widetilde{x}_0$ to $\widetilde{x}_1$ is exactly a based isomorphism between the coverings $(\widetilde{X}, \widetilde{x}_0)$ and $(\widetilde{X}, \widetilde{x}_1)$. By Proposition 6.2, we know that a based isomorphism exists iff the associated subgroups are equal, i.e., when

$$g^{-1}Hg = H.$$

Let

$$D_g : (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, \widetilde{x}_1)$$

be this unique based isomorphism.

The elements $g \in G$ satisfying $g^{-1}Hg = H$ form a subgroup of G, called the **normalizer** of H:

$$N_G(H) = \{g \in G : g^{-1}Hg = H\}.$$

It contains H, and H is normal in $N_G(H)$. Thus $N_G(H)/H$ is a quotient group, even when H is not normal in the whole G.

Now we can compute the deck transformation group $\mathrm{Deck}(p)$:

Theorem 6.8. *For a connected covering $p : \widetilde{X} \to X$ with associated subgroup $H \subseteq \pi_1(X) =: G$,*

$$\mathrm{Deck}(p) \cong N_G(H)/H.$$

Proof. The isomorphism is given by $g \mapsto D_g$.

We skip the proof that this correspondence is a homomorphism and only check its kernel.

Suppose $D_g : (\widetilde{X}, \widetilde{x}_0) \to (\widetilde{X}, \widetilde{x}_1)$ is the identity. Then $\widetilde{x}_0 = \widetilde{x}_1$. By construction of D_g, the loop in X corresponding to g lifts to a path between $\widetilde{x}_0$ and $\widetilde{x}_1$. Since $\widetilde{x}_0 = \widetilde{x}_1$, this path is closed. Hence, g is in the associated subgroup of the covering, $g \in H$. □

If for *every* point of $p^{-1}(x_0)$ there is a deck transformation mapping $\widetilde{x}_0$ to this point then the covering is called **normal**:

Definition 6.9. A covering is **normal** (or **regular**) if, for every $x_0 \in X$ and any two points of $p^{-1}(x_0)$, there is a deck transformation taking the first point to the second.

The arguments above also characterize normal coverings:

Corollary 6.10. *A connected covering is normal if and only if H is a normal subgroup of G. In this case* $\mathrm{Deck}(p) \cong G/H$. *In particular, the universal cover is normal and its deck transformation group is isomorphic to G.*

5 All connected coverings of the circle

Every subgroup of $\pi_1(S^1, 1) \cong \mathbb{Z}$ is $n\mathbb{Z}$ for a unique $n \geq 0$. Since $\mathbb{Z}$ is abelian, conjugation does not change these subgroups.

For $n \geq 1$, the covering

$$p_n : S^1 \longrightarrow S^1, \qquad p_n(z) = z^n,$$

has associated subgroup $n\mathbb{Z}$. It is therefore the unique connected covering with that subgroup. Its sheet number is $[\mathbb{Z} : n\mathbb{Z}] = n$, and it is normal, with

$$\mathrm{Deck}(p_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

What are these deck transformations? The fiber over 1, $p_n^{-1}(1)$, is the set of the nth roots of unity, $e^{2\pi ik/n}$, $\quad k = 0, \ldots, n-1$. We know that for each of these roots there is a unique deck transformations sending 1 to this root and there are no other deck transformations. Explicitly, they are given by

$$D_k(z) = e^{2\pi ik/n} z, \qquad k = 0, \ldots, n-1.$$

For $n = 0$, the subgroup is $\{0\}$. The corresponding covering is the universal cover

$$p : \mathbb{R} \longrightarrow S^1, \qquad p(u) = e^{2\pi iu}.$$

It has infinitely many sheets, and $\mathrm{Deck}(p) \cong \mathbb{Z}$. Its deck transformations are exactly $T_k(u) = u+k$, $k \in \mathbb{Z}$: these translations preserve p and take 0 to every point of the fiber $p^{-1}(1) = \mathbb{Z}$. Again, uniqueness from one point proves that the list is complete.

Thus p_n for $n \geq 1$ and $\mathbb{R} \to S^1$ is the complete list of the connected coverings of the circle.

Reference. Hatcher, *Algebraic Topology*, Section 1.3, p. 61 and pp. 66–71; Propositions 1.32, 1.36, 1.37, Theorem 1.38 and Proposition 1.39.