Lecture 5
Covering spaces

Lifting a loop at 1 in S^1 along $p : \mathbb{R} \to S^1$, $p(u) = e^{2\pi i u}$, from the initial point $0 \in \mathbb{R}$ gives a path in the line. That path need not close: its endpoint records the winding number. Covering spaces generalize this picture. We will see that the loops which do close upstairs form a subgroup of the fundamental group downstairs.

1 Coverings and their sheets

Under $p : \mathbb{R} \to S^1$, a small open arc in S^1 has infinitely many disjoint copies in $\mathbb{R}$. Each copy maps homeomorphically onto the arc. The notion of a covering map generalizes this local picture to more general spaces than $\mathbb{R}$,S^1.

Definition 5.1. A map $p : \widetilde{X} \to X$ is a **covering map** if every $x \in X$ has an open neighborhood U such that

$$p^{-1}(U) = \coprod_{\lambda} \widetilde{U}_\lambda, \qquad p|_{\widetilde{U}_\lambda} : \widetilde{U}_\lambda \longrightarrow U \text{ is a homeomorphism for every } \lambda.$$

This says that $p^{-1}(U)$ is a disjoint union of some U_λ ($\coprod$ denoting a disjoint union instead of the usual union $\bigcup$) and each U_λ is homeomorphically projected to U by p.

The space $\widetilde{X}$, together with p, is a **covering space** of X.

Such a neighborhood U is called **evenly covered**. The sets $\widetilde{U}_\lambda$ are its **sheets**. The **fiber over** x is the set $p^{-1}(x)$ of points lying above x. Each sheet contains exactly one point above each point of U, as in Figure 1.

The number of points in the fiber is locally constant, and therefore constant when X is connected. We call a covering n**-sheeted** when this number is n. Following Hatcher, $n = 0$ is allowed; a nonempty covering of a connected space is automatically surjective.

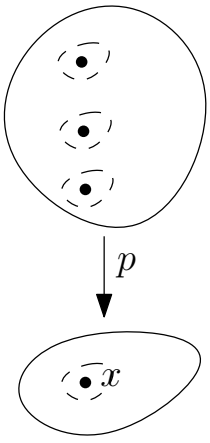


Figure 1: An evenly covered neighborhood and its sheets. Each sheet contains one point of the fiber over x.

Example 5.2 (Coverings of the circle)**.** The exponential map $p : \mathbb{R} \to S^1$ is an infinite-sheeted covering; the fiber over 1 is $\mathbb{Z}$. For $n \geq 1$, the map

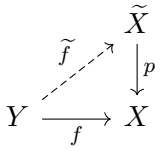
$$p_n : S^1 \longrightarrow S^1, \qquad p_n(z) = z^n,$$

is an n-sheeted covering. The inverse image of a small open arc consists of n disjoint open arcs, each mapped homeomorphically onto it. In particular, $p_n^{-1}(1)$ consists of the nth roots of unity.

A covering space need not be connected. For example, n disjoint copies of S^1, each mapping by the identity to S^1, also give an n-sheeted covering. This is different from p_n when $n > 1$.

2 Lifting paths and homotopies

Given a covering $p : \widetilde{X} \to X$ and a map $f : Y \to X$, a **lift** of f is a map $\widetilde{f} : Y \to \widetilde{X}$ such that $p\widetilde{f} = f$, in other words, the following diagram is commutative:



Thus a lift chooses a point $\widetilde{f}(y)$ above $f(y)$ for each y, continuously. It need not exist for an arbitrary Y. For a path, however, choosing its initial point upstairs is enough.

Proposition 5.3 (Path lifting)**.** *Let $p : \widetilde{X} \to X$ be a covering, let $\alpha : I \to X$ be a path, and choose $\widetilde{x}_0 \in p^{-1}(\alpha(0))$. There is a unique lift $\widetilde{\alpha} : I \to \widetilde{X}$ with $\widetilde{\alpha}(0) = \widetilde{x}_0$.*

Proof. See Figure 2. First suppose the whole path lies in an evenly covered neighborhood U. There is a unique sheet $\widetilde{U}$ containing $\widetilde{x}_0$. The lift is forced to be

$$\widetilde{\alpha} = (p|_{\widetilde{U}})^{-1}\alpha.$$

Indeed, a lift cannot pass from one sheet to another while it lies above U, since the sheets are disjoint open sets.

For a general path, subdivide it finitely many closed subintervals, each lying in its own evenly covered neighborhood. Such a subdivision exists by compactness, just as in the surjectivity proof of van Kampen's theorem. Lift the first piece from $\widetilde{x}_0$, the second from the endpoint of the first, and continue. The pieces agree at their endpoints and so give a continuous path. Each piece is forced by the previous endpoint, so the whole lift is unique. □

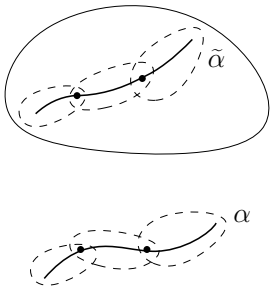


Figure 2: Lifting a path.

Example 5.4. For the n-sheeted covering $p_n : S^1 \to S^1$ with $p_n(z) = z^n$, the loop $\alpha(s) = e^{2\pi is}$ has lift $\widetilde{\alpha}(s) = e^{2\pi is/n}$ starting at 1. It ends at $e^{2\pi i/n}$, not at 1 when $n > 1$. Going n times around the base circle is needed to return to the starting point upstairs.

Consider now a homotopy $H_t : Y \to X$. Can we lift it? For any point y the homotopy gives a path $\alpha_y : I \to X$ with $t \mapsto H_t(y)$. So, in a way, we can think of H as a continuous family of paths $\{\alpha_y\}$ parametrized by $y \in Y$. Suppose for each of these paths we are given a lift of its starting point continuously depending on y. Then using the previous proposition there is a unique lift of every path. It turns out, that this unique lift is continuous in y and so it gives a lift of the whole H:

Theorem 5.5 (Homotopy lifting)**.** *Let $p : \widetilde{X} \to X$ be a covering and $H_t : Y \to X$ a homotopy. Given a lift $\widetilde{H}_0 : Y \to \widetilde{X}$ of H_0, there is a unique lifted homotopy $\widetilde{H}_t : Y \to \widetilde{X}$ starting at $\widetilde{H}_0$ and satisfying*

$$p\widetilde{H}_t = H_t \qquad (t \in I).$$

We use this theorem without proof. We have already explained how $\widetilde{H}$ is constructed; the additional work is to show that it is continuous. See Hatcher, Proposition 1.30 and the lifting argument on pp. 30–31.

Consider a special case of a homotopy of paths fixing the endpoints, i.e., $Y = I$ and the endpoints $a := H_t(0)$ and $b := H_t(1)$ stay the same for all t. On one hand the lifts of the endpoints, $\widetilde{H}_t(0)$ and $\widetilde{H}_t(1)$, vary continuously with t; on the other hand the stay in the discrete fibers $p^{-1}(a)$ and $p^{-1}(b)$. Hence, they don't move with t at all. Consequently, homotopic paths with the same endpoints, lifted from the same initial point, have the same final point. We will use this consequence repeatedly.

3 The subgroup associated to a covering

Choose $\widetilde{x}_0 \in \widetilde{X}$ and put $x_0 = p(\widetilde{x}_0)$. A loop at $\widetilde{x}_0$ projects to a loop at x_0, so we have the induced map

$$p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \longrightarrow \pi_1(X, x_0).$$

Unlike the map induced by an arbitrary continuous map, this homomorphism never kills a nontrivial loop class.

Proposition 5.6. *The homomorphism p_* is injective so it identifies $\pi_1(\widetilde{X}, \widetilde{x}_0)$ with a subgroup of $\pi_1(X, x_0)$. This subgroup consists exactly of the classes of loops downstairs which lift to closed loops (and not just paths) upstairs: their lifts starting at $\widetilde{x}_0$ must end also at $\widetilde{x}_0$.*

Proof. Suppose a loop $\widetilde{\alpha}$ upstairs projects to a nullhomotopic loop downstairs. Lift a homotopy from $p\widetilde{\alpha}$ to the constant loop, starting with $\widetilde{\alpha}$. This gives a homotopy from $\widetilde{\alpha}$ to the lift of a constant loop. But the constant loop lifts to the constant loop, so $\widetilde{\alpha}$ is nullhomotopic. This proves injectivity.

Let us prove the second part of the statement. If the lift of a loop α is a closed loop $\widetilde{\alpha}$, then $[\alpha] = p_*[\widetilde{\alpha}]$ belongs to the image of p_*.

Conversely, suppose $[\alpha]$ is in the image of p_*. We need to prove that α lifts to a closed loop. Since $[\alpha]$ is in the image, there is a loop upstairs $\widetilde{\alpha}'$ such that $p_*[\widetilde{\alpha}'] = [\alpha]$. So, there is a loop $\alpha' := p\widetilde{\alpha}'$ downstairs which is homotopic to α and which lifts to the loop $\widetilde{\alpha}'$ upstairs. During the homotopy between α' and α the endpoints stay fixed. As we have seen, this means that when we lift the homotopy the endpoints upstairs also stay fixed. So, since α' lifts to a loop, i.e., to a path whose endpoints are the same, then so does α. □

Example 5.7. Recall that $\pi_1(S^1, 1) \cong \mathbb{Z}$, with the isomorphism given by the winding number. Consider again the n-sheeted covering $p_n : S^1 \to S^1$ with $p_n(z) = z^n$.

The subgroup associated to p_n with basepoint 1 upstairs is $n\mathbb{Z} \subset \mathbb{Z}$. Indeed, the standard loop of winding number k, $z \mapsto z^k$, lifts from 1 to $s \mapsto e^{2\pi iks/n}$. It closes exactly when n divides k.

For $p : \mathbb{R} \to S^1$, the associated subgroup is $\{0\} \subset \mathbb{Z}$: the lift from $0 \in \mathbb{R}$ of a loop of winding number k ends at k; so it closes only when $k = 0$.

4 Universal covering spaces

A covering $p : \widetilde{X} \to X$ with $\widetilde{X}$ simply connected is called a **universal covering** of X. By Proposition 5.6, its associated subgroup is trivial.

Example 5.8. The covering $p : \mathbb{R} \to S^1$ is universal since $\mathbb{R}$ is simply connected.

To prove that a universal covering exists we need two technical assumptions on X. CW complexes and almost all other spaces we consider satisfy them.

Definition 5.9. A space X is **semilocally simply connected** if every $x \in X$ has an open neighborhood U such that every loop in U based at x is nullhomotopic in X. Note, that U does not have to be simply connected – the loops in U need to be contractible in X, not in U itself.

Definition 5.10. A space is **locally path connected** if, for every point y and every open neighborhood W of y, there is a path-connected open neighborhood V with $y \in V \subseteq W$.

Theorem 5.11 (Existence of a universal cover)**.** *A path-connected, locally path-connected, semilocally simply connected space has a universal covering.*

Proof. We will describe the construction and check some of its properties.

Fix $x_0 \in X$. Take another point $x_1 \in X$, possibly $x_1 = x_0$. Consider the set of all paths γ in X connecting x_0 to x_1. Generally, there are too many of them. We can simplify this set by considering the homotopy classes of such paths, with the homotopy fixing both ends, x_0 and x_1. This set of homotopy classes forms the fiber of a universal covering over x_1. Here is a precise definition. Set

$$\widetilde{X} = \{[\gamma] : \gamma : I \to X,\ \gamma(0) = x_0\}, \qquad p([\gamma]) = \gamma(1).$$

The covering map $p : \widetilde{X} \to X$ works in the obvious way: given a homotopy class $[\gamma]$ of some path in X it sends this class to the last point in the path, $\gamma(1)$.

We now describe the topology of $\widetilde{X}$. Take $[\gamma] \in \widetilde{X}$, where γ is a path between x_0 and some $x_1 \in X$. Let U be a neighborhood of x_1 which is path-connected and such that every loop in U is nullhomotopic in X. The corresponding neighborhood $\widetilde{U}$ of $[\gamma]$ consists of all classes $[\gamma\eta]$, where η is a path in U starting at x_1.

Let us check that $p : \widetilde{U} \to U$ is a one-to-one map. Indeed, suppose η, η' are two different paths in U both starting at x_1 and ending at x_2. Then $p[\gamma\eta] = [\gamma\eta'] = x_2$. The loop $\eta\overline{\eta}'$ lies in U and hence it is nullhomotopic in X. So, $\eta \simeq \eta'$ and $\gamma\eta \simeq \gamma\eta'$ meaning that $[\gamma\eta]$ and $[\gamma\eta']$ are the same points of $\widetilde{X}$.

Finally we check that $\widetilde{X}$ is simply connected. Suppose α is a loop in X. What is its lift? For each point $\alpha(t)$ of this loop let α_t be the part of α between $\alpha(0)$ and $\alpha(t)$. It is easy to see that for every t the point $\alpha(t)$ lifts to $[\alpha_t]$: this lift is continuous (which we do not check), and it is indeed a lift – $p[\alpha_t]$ is the last point of α_t, hence $\alpha(t)$.

So, α lifts to a path between $[\alpha_0] \in \widetilde{X}$ and $[\alpha_1] \in \widetilde{X}$. This path is closed when $[\alpha_0] = [\alpha_1]$. But α_0 is the constant loop at x_0 and $\alpha_1 = \alpha$, so $[\alpha_0] = [\alpha_1]$ only when α is nullhomotopic.

We have shown that α lifts to a closed loop only when α is nullhomotopic. But $p_*\pi_1(\widetilde{X})$ consists precisely of loops which lift to closed loops, hence the image of p_* is the trivial group. Since p_* is injective we get that $\pi_1(\widetilde{X})$ is trivial. □

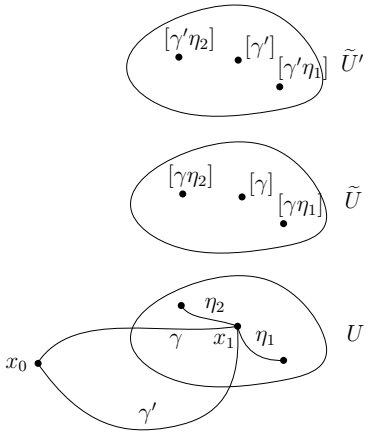


Figure 3: Universal covering of X. Paths γ and γ' are not homotopic and hence lift to different points in $\widetilde{X}$.

5 When does a map lift?

Suppose we are given a map $f : (Y, y_0) \to (X, x_0)$ and we want to lift it to a map $\widetilde{f} : Y \to \widetilde{X}$. Choose some $\widetilde{x}_0 \in p^{-1}(x_0)$ and set $\widetilde{f}(y_0) = \widetilde{x}_0$. How do we know the rest of the values of $\widetilde{f}$?

Let $y_1 \in Y$ be a point. Connect it by a path γ with y_0. As we know from before, $f\gamma : I \to X$ lifts to a unique path in $\widetilde{X}$, so the only way we can proceed is to assign $\widetilde{f}(y)$ to be the endpoint of this lift. But what if some other path γ' from y_0 to y lifts with a different endpoint? Then we will have to assign two different values to $\widetilde{f}(y)$, which is impossible.

Theorem 5.12 (Lifting criterion)**.** *Let $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a covering and $f : (Y, y_0) \to (X, x_0)$ a based map. Suppose Y is path connected and locally path connected. A lift $\widetilde{f}$ with $\widetilde{f}(y_0) = \widetilde{x}_0$ exists if and only if*

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)).$$

Such a lift is unique.

Proof. If $f = p\widetilde{f}$, then $f_* = p_*\widetilde{f}_*$, which gives the required inclusion.

Conversely, assume the inclusion. For $y \in Y$, choose a path γ from y_0 to y and define $\widetilde{f}(y)$ to be the endpoint of the lift of $f\gamma$ from $\widetilde{x}_0$. We are only going to check that this construction is well defined, i.e., does not depend on the choice of γ.

For another choice γ', the loop $f(\gamma \cdot \overline{\gamma'})$ represents an element of the image of f_*, hence of the image of p_*. It lifts to a closed loop by Proposition 5.6.

The loop $f(\gamma \cdot \overline{\gamma'})$ is the concatenation $f(\gamma) \cdot f(\overline{\gamma'})$. So it lifts to the concatenation $\widetilde{\alpha} \cdot \widetilde{\alpha}'$, where $\widetilde{\alpha}$ is the lift of $f(\gamma)$ starting at $\widetilde{x_0}$ and $\widetilde{\alpha}'$ is the lift of $f(\overline{\gamma'})$ starting at the end of $\widetilde{\alpha}$. Since this lift closes we get that the endpoint of $\widetilde{\alpha}'$ is x_0. Reversing $\widetilde{\alpha}'$ we see that it is the lift of $f(\gamma')$ starting at x_0. Its endpoint is the starting point of $\widetilde{\alpha}'$, i.e., the ending point of $\widetilde{\alpha}$. So, the lifts of both $f(\gamma)$ and $f(\gamma')$ starting at x_0 have the same endpoint, the endpoint of $\widetilde{\alpha}$. □

Reference. Hatcher, *Algebraic Topology*, Section 1.3, pp. 56–57 and 60–65; Propositions 1.30, 1.31 and 1.33.