Lecture 4
Van Kampen's theorem

Knowing the fundamental groups of two spaces should help us compute the fundamental group of a space obtained by gluing them together. Van Kampen's theorem makes this possible: it tells us which loops generate the new group and which relations the gluing imposes.

From now on we may implicitly identify loops with their homotopy classes relative to the endpoints. So, a word "loop" might be a shortcut for its homotopy class. In case we need to distinguish these two notions to avoid confusion we will be explicit about it.

1 Loops as words

Consider two circles meeting at one point x_0, the space $S^1 \vee S^1$. Let a go once around the first circle and b once around the second. A word such as $a^2b^{-1}a$ describes a loop: go twice around the first circle, once backwards around the second, and once around the first again. Backtracking cancels, so, for example,

$$(a^2b^{-1})(ba^{-1}) = a.$$

Is every loop described by such a word? Can two words represent homotopic loops even when you cannot get from one to another using elementary canceling such as $a^{-1}a = 1$, e.g., is aba the same as bab? We will answer both questions using van Kampen's theorem. First we describe the group of words itself.

Definition 4.1. The **free product** $G*H$ of two groups consists of the **reduced words** $g_1 \cdots g_m$. Here "reduced" means that each letter is a nonidentity element of G or H, and successive letters belong to different groups. The empty word is allowed.

To multiply two words, concatenate them, multiply adjacent letters from the same group, and delete identity letters until the word is reduced. The empty word is the identity, and the inverse of $g_1 \cdots g_m$ is $g_m^{-1} \cdots g_1^{-1}$. These operations make $G * H$ a group, containing copies of G and H as the one-letter words together with the identity.

In particular, $\mathbb{Z} * \mathbb{Z}$ is the group of reduced words in two generators a, b and their inverses. The words ab and ba are different, so this group is not abelian. More generally, the **free group** F_r on r generators is the free product of r copies of $\mathbb{Z}$.

Homomorphisms $u : G \to K$ and $v : H \to K$ determine a unique homomorphism $G * H \to K$: replace every letter by its image and multiply in K. Reducing a word does not change this product, because u and v are homomorphisms. We will use this to send words of loops to their concatenations.

2 Van Kampen's theorem

Suppose $X = U \cup V$, where U, V are open and $U, V, U \cap V$ are path connected. Choose $x_0 \in U \cap V$. A loop in U or in V is also a loop in X, so the inclusions give a homomorphisms $\pi_1(U, x_0) \to \pi_1(X, x_0)$, $\pi_1(V, x_0) \to \pi_1(X, x_0)$. They in turn give a homomorphism

$$\Phi : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0).$$

It sends a word of loops to the concatenation of these loops.

There is an immediate reason why Φ might not be injective: take a loop in $U \cap V$, it corresponds to different letters in $\pi_1(U, x_0) * \pi_1(V, x_0)$ but is the same loop in $\pi_1(X, x_0)$. Write

$$i_U : U \cap V \hookrightarrow U, \qquad i_V : U \cap V \hookrightarrow V.$$

For every $\omega \in \pi_1(U \cap V, x_0)$, we must therefore impose the relation $(i_U)_*(\omega) = (i_V)_*(\omega)$.

Imposing a relation $r = 1$ in a group also makes every conjugate grg^{-1} trivial. For a subset R of a group, its **normal closure** $\langle\langle R \rangle\rangle$ is the smallest normal subgroup containing R. Its elements are products of conjugates of elements of R and their inverses. Thus quotienting by $\langle\langle R \rangle\rangle$ imposes exactly the relations $r = 1$ for $r \in R$ and their consequences.

Theorem 4.2 (Van Kampen)**.** *Let $X = U \cup V$, with U, V open and $U, V, U \cap V$ path connected, and let $x_0 \in U \cap V$. Then Φ is surjective and its kernel is*

$$N = \langle\langle (i_U)_*(\omega)(i_V)_*(\omega)^{-1} : \omega \in \pi_1(U \cap V, x_0) \rangle\rangle.$$

Consequently, Φ induces an isomorphism

$$\pi_1(X, x_0) \cong \big(\pi_1(U, x_0) * \pi_1(V, x_0)\big)/N.$$

In words, the theorem gives a complete descriptions of $\pi_1(U \cap V, x_0)$. This group consists of all the reduced words with letters (loops) from $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$. Additionally, we need to impose one obvious relation: given any loop ω in $U \cap V$ the letters corresponding to it in $\pi_1(U, x_0)$ and in $\pi_1(V, x_0)$ have to be identified, i.e., considered equal. No further relations are needed.

Surjectivity of Φ

Let α be a loop in X at x_0. Subdivide its parameter interval so that each piece lies entirely in U or entirely in V. Such a finite subdivision exists because $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ form an open cover of the compact interval: all sufficiently short subintervals lie in one member of this cover. Combine consecutive pieces in the same set. We obtain $\alpha = \alpha_1 \cdots \alpha_m$ with pieces alternating between U and V.

The pieces α_i need not be loops. The common endpoint of α_i and α_{i+1} is in $U \cap V$ (because one of α_i, α_{i+1} is in U and another is in V). So, we may choose a path h_i in $U \cap V$ connecting x_0 to this common endpoint. At the two ends take $h_0 = h_m$ to be the constant path staying at x_0. Now

$$\ell_i = h_{i-1} \cdot \alpha_i \cdot \overline{h_i}$$

is a closed loop in U, if α_i was in U, and in V, if α_i was in V. In the concatenation $\ell_1 \cdots \ell_m$, each inserted journey $\overline{h_i} \cdot h_i$ cancels. Hence

$$\alpha = \ell_1 \cdots \ell_m$$

is a product of loops in U and V. This proves surjectivity, see Figure 1.

There are no further relations

You can find a proof in Hatcher, *Algebraic Topology*, `https://pi.math.cornell.edu/~hatcher/`, pp. 44–46.

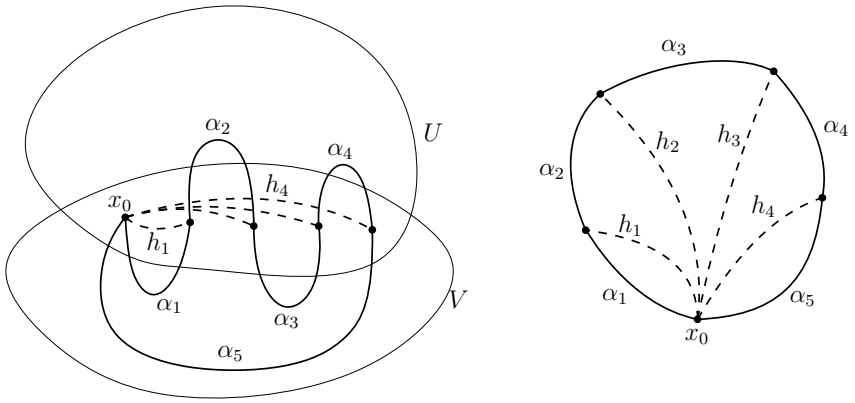


Figure 1: Proof of surjectivity.

3 Circles and spheres

Example 4.3 (A wedge of circles)**.** Return to $X = S^1 \vee S^1$. The two circle summands are not open in X, so we thicken them slightly, as in Figure **??**. Let U be the first circle together with a small open arc of the second around x_0; likewise, V be the second circle together with a small open arc of the first around x_0;. Then U and V are open in X, each deformation retracts onto its circle, and $U \cap V$ is a contractible cross-shaped neighborhood of x_0.

Any loop in $U \cap V$ is null-homotopic in $U \cap V$ and hence in U. So, it represents the unit of $\pi_1(U)$, the trivial word 1. Likewise, any loop in $U \cap V$ represents the unit of $\pi_1(V)$. So, the only relation we need to add is the trivial one $1 = 1$.

Therefore

$$\pi_1(S^1 \vee S^1, x_0) \cong \mathbb{Z} * \mathbb{Z} = F_2.$$

This answers our opening questions: every loop gives a word in a, b, and two reduced words give the same loop class only when they are identical. In particular, $aba \neq bab$. Repeating the same open-neighborhood argument gives

$$\pi_1(\underbrace{S^1 \vee \cdots \vee S^1}_{r \text{ circles}}) \cong F_r.$$

Example 4.4 (Higher spheres)**.** Let $n \geq 2$. Cover S^n by $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$, where N, S are the north and south poles. Each is homeomorphic to $\mathbb{R}^n$, hence contractible. Their intersection deformation retracts onto the equator S^{n-1} and is path connected. Van Kampen gives

$$\pi_1(S^n) = 1 \qquad (n \geq 2).$$

Notice that we only needed the intersection to be path connected, not simply connected. For S^1 the analogous intersection has two components, so the theorem does not apply. This is why covering a circle by two contractible arcs does not prove that its group is trivial.

4 Attaching cells introduces relations

Suppose we attach a disk to a path-connected CW complex X along a map $\varphi : S^1 \to X$, obtaining $X \cup_\varphi D^2$. Choose $x_0 \in \varphi(S^1)$. How does $\pi_1(X, x_0)$ change when we attach the disk?

For now choose $x_0 = \varphi(1)$. Following the boundary of the disk once gives a loop φ at x_0. After the disk is attached, this loop bounds a disk, so it becomes trivial in $\pi_1(Y, x_0)$. So, its normal closure $\langle\langle[\varphi]\rangle\rangle$ becomes trivial. From Van Kampen it follows that this is exactly what changes:

Proposition 4.5. *The inclusion $X \hookrightarrow X \cup_\varphi D^2$ induces a surjective homomorphism with kernel $\langle\langle[\varphi]\rangle\rangle$. Thus*

$$\pi_1(X \cup_\varphi D^2, x_0) \cong \pi_1(X, x_0)/\langle\langle[\varphi]\rangle\rangle.$$

Proof. The map φ does no have to be one-to-one on S^1. To deal with that consider a slightly smaller disk inside D^2, thus D^2 is split into a smaller disk and a collar. The boundary of the smaller disk is attached by a map φ' which is homotopic to φ and which is one-to-one. So, instead of attaching D^2 to X we are going to first attach the collar and get X' and then attach the smaller disk to X' by a one-to-one map φ'. Because X' deformation retracts to X we might simply rename X' and φ' to X and φ and from now on assume that φ is one-to-one, see Figure **??**.

Let U be a small open neighborhood of X in $X \cup_\varphi D^2$ and let V be a small open neighborhood of $\varphi(D^2)$ in $X \cup_\varphi D^2$.

We have that U deformation retracts to X. So, $\pi_1(U, x_0)$ is isomorphic to $\pi_1(X, x_0)$ and every loop in U is homotopic to some loop in X. Same goes for V and $\varphi(D^2)$. Finally, $U \cap V$ deformation retracts to $\varphi(S^1)$. This is again a purely technical trick we need in orderto apply Van Kampen – here we deal with the issue that X and $\varphi(D^2)$ aren't open.

Because φ is one-to-one, the fundamental group of $U \cap V$ is the same as the fundamental group of S^1, so it is $\mathbb{Z}$ generated by the loop $\varphi : S^1 \to U \cap V$. The fundamental group of U is $\pi_1(X)$ and the fundamental group of V is that of the 2-disk, i.e., trivial.

Van Kampen now tells us that to obtain $X \cup_\varphi D^2$ we need to first take the free product of $\pi_1(X)$ with the trivial group $\pi_1(D^2)$, this is just $\pi_1(X)$. Then we need identify every loop in $U \cap V$ considered as a letter in $\pi_1(X)$ with the same loop considered as a letter in $\pi_1(D^2)$. Every loop in $U \cap V$ is generated by φ, when considered as a loop in X it is simply $[\varphi]$ and when considered as a loop in D^2 it is trivial.

So, the relation we need to impose in $\pi_1(X)$ is exactly $[\varphi] = 1$, which is equivalent to taking the quotient $\pi_1(X, x_0)/\langle\langle[\varphi]\rangle\rangle$. □

Corollary 4.6. *Attaching an n-cell for $n \geq 3$ to a path-connected CW complex does not change its fundamental group: the inclusion induces an isomorphism.*

Proof. Use the same argument. The difference now is that $\pi_1(\partial D^n) = 1$ when $n \geq 3$. Hence, every loop in $U \cap V \simeq \partial D^n$ is already nullhomotopic there and attaching an n-cell imposes no additional relations. □

Presentations and the torus

Start with r circles joined at one vertex and attach finitely many 2-cells. Each attaching loop gives a word w_j in the circle generators $a_1, \dots, a_r$. Applying Proposition 4.5 one cell at a time gives

$$\pi_1(X) \cong F_r / \langle\langle w_1, \dots, w_k \rangle\rangle = \langle a_1, \dots, a_r \mid w_1, \dots, w_k \rangle.$$

The expression on the right is a **group presentation**: the symbols before the bar are generators, and each word after the bar is required to equal 1. Thus the 1-cells supply generators and the 2-cells supply relations. Attaching finitely many higher cells leaves this group unchanged.

Example 4.7 (The torus)**.** Represent the torus by a square with opposite sides identified. Its four vertices become one vertex, its edges become two circles a, b, and its interior is one 2-cell. Orient both horizontal edges to the right and both vertical edges upwards. Reading the boundary counterclockwise from the bottom left corner gives

$$a\, b\, a^{-1}\, b^{-1}.$$

Consequently,

$$\pi_1(T^2) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle.$$

The relation says $ab = ba$. Hence every element can be written $a^m b^n$. There are no further identifications between these expressions: sending a to $(1, 0)$ and b to $(0, 1)$ defines a homomorphism to $\mathbb{Z}^2$, since the relator maps to $(0, 0)$. It sends $a^m b^n$ to (m, n). Thus

$$\pi_1(T^2) \cong \mathbb{Z}^2.$$

Reference. Hatcher, *Algebraic Topology*, `https://pi.math.cornell.edu/~hatcher/`, Section 1.2, pp. 40–46 and 49–51.