Lecture 1
Homotopy and homotopy type

Conventions. All maps between topological spaces are continuous. Write $I = [0, 1]$ for the unit segment, and let id_X denote the identity map of X. Composition of maps is written $gf = g \circ f$: first apply f, then g.

1 Homotopies of maps

Definition 1.1. Let $f, g : X \to Y$ be maps. A **homotopy from f to g** is a continuous map

$$H : X \times I \longrightarrow Y \quad \text{such that} \quad H(x, 0) = f(x), \qquad H(x, 1) = g(x)$$

for every $x \in X$. If such a map exists, we say that f and g are **homotopic**, and write $f \simeq g$.

For a fixed $t \in I$, write $H_t(x) = H(x, t)$. Thus $H_0 = f$ and $H_1 = g$. For a fixed x, the map $t \mapsto H_(x)$ is the path followed by the image of x. Continuity is required jointly in x and t; specifying a continuous map H_t for each t does not by itself give a homotopy.

Example 1.2 (Straight-line homotopies)**.** Let $C \subseteq \mathbb{R}^n$ be convex: the line segment between any two points of C lies in C. Any two maps $f, g : X \to C$ are homotopic by the formula

$$H_t(x) = (1 - t)f(x) + tg(x).$$

The values of H lie in C by convexity, and substituting $t = 0, 1$ gives the required endpoint maps.

Definition 1.3. Let $A \subseteq X$, and suppose that $f, g : X \to Y$ agree on A. A homotopy H from f to g is **relative to** A if

$$H_t(a) = f(a) = g(a) \qquad (a \in A,\ t \in I).$$

We then write $f \simeq g \ \mathrm{rel}\, A$.

The images of points of A remain fixed throughout the homotopy.

A **path** in Y is a map $\alpha : I \to Y$. A homotopy between paths $\alpha, \beta : I \to Y$ **fixing endpoints** means a homotopy relative to $\{0, 1\}$, i.e.:

$$H_t(0) = \alpha(0) = \beta(0), \qquad H_t(1) = \alpha(1) = \beta(1)$$

for every t. In a convex subset of $\mathbb{R}^n$, two paths with the same endpoints are homotopic fixing endpoints by Example 1.2.

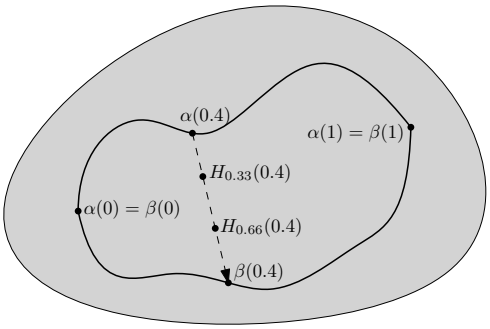


Figure 1: A homotopy H relative to the endpoints between two paths $\alpha, \beta : I \to C$ inside a convex set C.

Proposition 1.4. *Homotopy is an equivalence relation on the set of maps $X \to Y$. The same holds for homotopy relative to A, among maps with a fixed restriction to A.*

Proof. For reflexivity, use the constant-in-time homotopy $H_t(x) = f(x)$. If H is a homotopy from f to g, then $G_t = H_{1-t}$ is a homotopy from g to f, proving symmetry.

For transitivity, let H run from f to g, and let G run from g to h. Run each homotopy at twice its original speed:

$$L_t(x) = \begin{cases} H_{2t}(x), & 0 \le t \le \frac{1}{2}, \\ G_{2t-1}(x), & \frac{1}{2} \le t \le 1. \end{cases}$$

At $t = \frac{1}{2}$, both expressions equal $g(x)$.

If the original homotopies are relative to A, each of these constructions keeps the same fixed values on A. □

Proposition 1.5 (Compatibility with composition)**.** *Suppose $f \simeq g : X \to Y$. For maps $u : W \to X$ and $v : Y \to Z$, we have*

$$vfu \simeq vgu.$$

Proof. If H is the given homotopy, use $F(w,t) = v(H(u(w),t))$. This is a composition of continuous maps and has the desired endpoints. □

For example, if $f_0 \simeq f_1 : X \to Y$ and $g_0 \simeq g_1 : Y \to Z$, then

$$g_0 f_0 \simeq g_0 f_1 \simeq g_1 f_1.$$

2 Homotopy equivalence

Definition 1.6. A map $f : X \to Y$ is a **homotopy equivalence** if there is a map $g : Y \to X$ such that

$$gf \simeq \mathrm{id}_X, \qquad fg \simeq \mathrm{id}_Y .$$

If a homotopy equivalence $X \to Y$ exists, we say that X and Y have the same **homotopy type**, and write $X \simeq Y$.

The notation has two uses: $f \simeq g$ compares maps with the same domain and codomain, while $X \simeq Y$ compares spaces. A homotopy equivalence need not be injective or surjective.

Proposition 1.7. *Having the same homotopy type is an equivalence relation on spaces. Every homeomorphism is a homotopy equivalence.*

Proof. The identity gives reflexivity. If f and g satisfy the definition, exchanging their roles gives symmetry.

For transitivity, suppose $f : X \to Y$ and $h : Y \to Z$ are homotopy equivalences, and let $g : Y \to X$ and $k : Z \to Y$ be the maps provided by the definition. We show that $hf : X \to Z$ and $gk : Z \to X$ satisfy the same definition. Composing the given homotopies with the appropriate maps gives

$$(gk)(hf) = g(kh)f \simeq g\,\mathrm{id}_Y\, f = gf \simeq \mathrm{id}_X\,.$$

Similarly,

$$(hf)(gk) = h(fg)k \simeq h\,\mathrm{id}_Y\, k = hk \simeq \mathrm{id}_Z\,.$$

Hence hf is a homotopy equivalence.

If $h : X \to Y$ is a homeomorphism, then h, h^{-1} satisfy the definition of a homotopy equivalence. □

3 Contractible spaces

Definition 1.8. A map is **nullhomotopic** if it is homotopic to a constant map. A nonempty space X is **contractible** if its identity map is nullhomotopic. Thus there are a point $p \in X$ and a homotopy

$$C : X \times I \to X, \qquad C_0(x) = x, \quad C_1(x) = p \quad \text{for all } x \in X$$

Such a homotopy is called a **contraction** to p.

Proposition 1.9. *A space is contractible if and only if it has the homotopy type of a point.*

Proof. Suppose C contracts X to p. Let $q : X \to \{*\}$ be the unique map, and let $j : \{*\} \to X$ send $*$ to p. Then $qj = \mathrm{id}_{\{*\}}$, while jq is the constant map with value p, so $jq \simeq \mathrm{id}_X$.

Conversely, if $X \simeq \{*\}$, there is a map $j : \{*\} \to X$ such that $\mathrm{id}_X \simeq jq$. This homotopy contracts X to $j(*)$. □

Corollary 1.10. *Every contractible space is path connected: any two of its points can be joined by a path.*

Proof. If C contracts X to p, then $t \mapsto C_t(x)$ is a path from any $x \in X$ to p. □

Example 1.11 (Convex sets)**.** Every nonempty convex subset $C \subseteq \mathbb{R}^n$ is contractible. Choose $p \in C$ and use

$$H_t(x) = (1 - t)x + tp.$$

In particular, Euclidean spaces, intervals, and closed balls are contractible. This is the same straight-line construction used for homotopies of maps.

In the definition of contractibility we do *not* require $C_t(p) = p$. That additional condition is part of a deformation retraction onto a point, defined below. Our convex-set contraction happens to satisfy it.

4 Retractions and deformation retractions

Definition 1.12. Let $A \subseteq X$, and let $i : A \to X$ be the inclusion map, $i(a) = a$. A **retraction** is a map $r : X \to A$ such that $r(a) = a$ for every $a \in A$, or equivalently $ri = \mathrm{id}_A$. If such a map exists, A is a **retract** of X.

A retraction specifies where points end up. It does not provide any continuous motion from a point x to its image $r(x)$.

Definition 1.13. A **deformation retraction of X onto A** is a homotopy $H : X \times I \to X$ such that

$$H_0(x) = x, \qquad H_1(x) \in A, \qquad H_t(a) = a \quad (x \in X,\ a \in A,\ t \in I).$$

Equivalently,

$$ir \simeq \mathrm{id}_X$$

Then A is called a **deformation retract** of X.

Proposition 1.14. *If A is a deformation retract of X, the inclusion $i : A \to X$ is a homotopy equivalence.*

Proof. We have $ri = \mathrm{id}_A$, and the deformation retraction gives $ir \simeq \mathrm{id}_X$. These are precisely the two required conditions. □

Question for the class. Can every retraction be obtained from a deformation retraction? Consider a space consisting of two isolated points.

Example 1.15 (A retraction need not be a deformation retraction)**.** Give $X = \{a, b\}$ the discrete topology and put $A = \{a\}$. The constant map $r : X \to A$ is a retraction. A deformation retraction would give a path $t \mapsto H(b, t)$ from b to a, which is impossible.

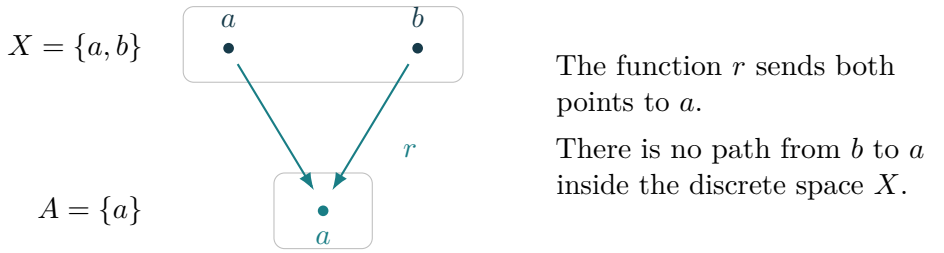


Figure 2: A retraction without a deformation retraction. The boxes only group the points into sets; the arrows describe a function between the sets, not paths inside X.

In particular, the existence of a retraction r does not imply that its inclusion map i is a homotopy equivalence.

5 Explicit deformation retractions

For $n \geq 1$, let $\|x\|$ denote the Euclidean norm of $x \in \mathbb{R}^n$, and define

$$D^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}, \qquad S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

To verify a proposed deformation retraction, check four things: joint continuity, the initial map, the final image, and the fact that every point of the target subspace remains fixed.

A ball onto a point

The homotopy $H_t(x) = (1-t)x$ deformation retracts D^n onto $\{0\}$. Hence $D^n \simeq \{0\}$.

A product onto one end

For any space X, the subspace $X \times \{0\}$ is a deformation retract of $X \times I$. Use

$$H_t(x, s) = (x, (1-t)s).$$

An annulus onto a circle

Let

$$A = \{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}.$$

Radial projection $r(x) = x/\|x\|$ is a retraction $A \to S^1$. The homotopy:

$$H_t(x) = (1-t)x + t\frac{x}{\|x\|} = \left(1 - t + \frac{t}{\|x\|}\right) x.$$

gives a deformation retraction.

Punctured Euclidean space onto a sphere

The same formula works on the larger space $X = \mathbb{R}^n \setminus \{0\}$, with target S^{n-1}:

$$H_t(x) = \left(1 - t + \frac{t}{\|x\|}\right) x.$$

When $\|x\| < 1$, a point moves outward toward the sphere; when $\|x\| > 1$, it moves inward. For $n = 1$ the target is $S^0 = \{-1, 1\}$. Therefore

$$\mathbb{R}^n \setminus \{0\} \simeq S^{n-1} \qquad (n \geq 1).$$

This is a statement about homotopy type, not homeomorphism: the sphere is compact and punctured Euclidean space is not. It also does not show that a sphere is contractible. For instance, the formula $H_t(x) = (1-t)x$ fails to contract punctured Euclidean space *inside itself*, since at $t = 1$ it reaches the deleted point.

6 Optional: mapping cylinders

The product example slides a cylinder onto one end. The mapping cylinder allows that end to be attached to another space by an arbitrary map.

Definition 1.16. For a map $f : X \to Y$, its **mapping cylinder** is the quotient space

$$M_f = \big((X \times I) \sqcup Y\big)/\big((x, 1) \sim f(x)\big).$$

Here $\sqcup$ denotes disjoint union.

In other words, the mapping cylinder M_f is obtained by gluing $X \times \{1\}$ to Y by the map f.

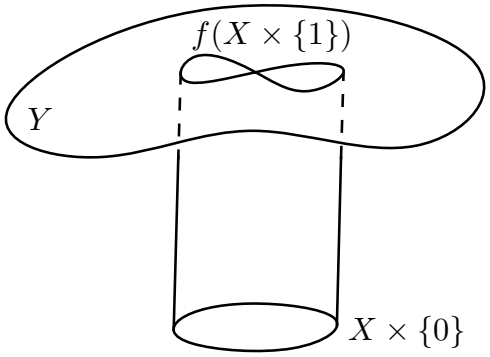


Figure 3: Mapping cylinder M_f.

Proposition 1.17. *The mapping cylinder M_f deformation retracts onto Y.*

Every point of the cylinder has a form $(x, s) \in X \times I$ or $y \in Y$.

The deformation moves each cylinder point toward its attached end and leaves Y fixed:

$$H_t(x, s) = (x, (1 - t)s + t), \qquad H_t(y) = y.$$

When $s = 1$, the first formula is $(x, 1) = f(x)$ for every t. It therefore agrees with the formula on Y, so the map is well-defined. At $t = 0$ it is the identity. At $t = 1$ every point lies in Y, since $(x, 1) = f(x)$, and it fixes Y at all times. This sliding is continuous.

In particular $M_f \simeq Y$, regardless of whether f is a homotopy equivalence. For $f = \mathrm{id}_X$, this is a cylinder sliding onto one end.

Reference. Hatcher, *Algebraic Topology*, Chapter 0, pp. 1–4.